ECON501 Advanced Microeconomic Theory 1
Fall Semester 2007
Problem Set 7

The due date for this problem set is Monday November 19

1. Assume a finite state space $\mathcal{S} = \{s_1, \ldots, s_N\}$ and a one-dimensional consequence space
$$\mathcal{X} = \mathbb{R}_+$$
Suppose a person's preference relation over the set of state-contingent vectors $\mathbf{c} = (c_1, \ldots, c_N) \in \mathbb{R}_+^N$, admits a Subjective Expected Utility Representation. That is, there exists a probability vector $(\pi_1, \ldots, \pi_N)$ and utility index $u : \mathbb{R}_+ \to \mathbb{R}$ such that for any pair of acts c, c':
$$\mathbf{c} \succcurlyeq \mathbf{c}' \Leftrightarrow \sum_{n=1}^{N} \pi_n u(c_n) \geq \sum_{n=1}^{N} \pi_n u(c'_n)$$
Furthermore, assume that $u(\cdot)$ is a strictly concave and strictly increasing function and that $\pi_n > 0$ for each $n = 1, \ldots, N$. Since $\mathcal{X} = \mathbb{R}_+$, we can define a convex combination of two contingent consumption vectors as follows: for any pair of contingent consumption vectors $\mathbf{c}, \mathbf{c}'$ and any α in $(0, 1)$, $\mathbf{c}'' = \alpha\mathbf{c} + (1 - \alpha)\mathbf{c}''$ is the contingent consumption vector for which
$$c''_n = \alpha c_n + (1 - \alpha) c'_n \text{ for every } n = 1, \ldots, N.$$
   (a) Show that if for any pair of contingent consumption vectors $\mathbf{c}$ and $\mathbf{c}'$
   $$\mathbf{c} \sim \mathbf{c}' \text{ and } \mathbf{c} \neq \mathbf{c}'$$
   then
   $$\mathbf{c}'' = \frac{1}{2}\mathbf{c} + \frac{1}{2}\mathbf{c}' \succ \mathbf{c}$$
   (b) Explain why is this not a contradiction of the independence axiom.

   Notation: for any pair of contingent consumption vectors $\mathbf{c}$ and $\mathbf{c}'$ and any event $E \subset \mathcal{S}$, let $\mathbf{c}_E\mathbf{c}'$ denote the contingent consumption vector $\mathbf{c}''$ for which
$$c''_n = \begin{cases} c_n \text{ if } s_n \in E \\ c'_n \text{ if } s_n \notin E \end{cases}$$

(c) Show that a preference relation that admits a subjective expected utility representation satisfies the following property called the *sure-thing principle.*

Sure-Thing Principle: For any two contingent consumption vectors $\mathbf{c}$ and $\mathbf{c}'$, and any event $E \subset \mathcal{S}$

$$\mathbf{c} \succsim \mathbf{c}'_E\mathbf{c} \Leftrightarrow \mathbf{c}_E\mathbf{c}' \succsim \mathbf{c}'.$$

Now consider a preference relation $\succsim$ that is strictly monotonic, continuous and satisfies the sure-thing principle. Debreu proved that such preference relation admits a *separable* representation, that is,

$$V(\mathbf{c}) = \sum_{n=1}^{N} v_n(c_n), \text{ where each } v_n(.) \text{ is increasing.}$$

Suppose further that each $v_n(.)$ is differentiable (so by strict monotonicity, we have $v_n'(c) > 0$ for all c and every $n = 1, \ldots, N$) and that the preference relation also satisfies the following property:

Certainty homotheticity. For any $\bar{c} > 0$, and for any pair of states s, t,

$$MRS_{mn}^{(\bar{c},\ldots,\bar{c})} = \frac{\partial V(\bar{c},\ldots,\bar{c})/\partial c_n}{\partial V(\bar{c},\ldots,\bar{c})/\partial c_m} = \frac{\partial V(\lambda\bar{c},\ldots,\lambda\bar{c})/\partial c_n}{\partial V(\lambda\bar{c},\ldots,\lambda\bar{c})/\partial c_m} = MRS_{mn}^{(\lambda\bar{c},\ldots,\lambda\bar{c})}$$

(d) Show that for our preference relation that is strictly monotonic, continuous and satisfies the sure-thing principle, *certainty homotheticity* implies that

$$\frac{v_n'(\bar{c})}{v_1'(\bar{c})} = \frac{v_n'(\lambda\bar{c})}{v_1'(\lambda\bar{c})} = \ell_n > 0,\ n = 2, \ldots, N$$

or equivalently

$$v_n'(c) = \ell_n v_1'(c) \text{ for all } c > 0,\ n = 2, \ldots, n \qquad (*)$$

(e) For each $n = 2, \ldots, N$, show that integrating both sides of $(*)$ yields

$$v_n(c) = \ell_n v_1(c) + \alpha_n$$

And so,

$$V(\mathbf{c}) = v_1(c_1) + \ell_2 v_1(c_2) + \ldots + \ \ell_n(c_n)\, v_1(c_2) + \sum_{n=2}^{N} \alpha_n$$

(f) Using your answer from (e) show that $\succsim$ admits a subjective expected utility representation. That is, we can find weights, $\pi_1, \ldots, \pi_N$ and a utility index $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, such that for any pair of contingent consumption vectors $\mathbf{c}$ and $\mathbf{c}'$,

$$\mathbf{c} \succcurlyeq \mathbf{c}' \Leftrightarrow \sum_{n=1}^{N} \pi_n u(c_n) \geq \sum_{n=1}^{N} \pi_n u(c'_n).$$

2. The purpose of this exercise is to illustrate how expected utility theory allows us to make consistent decisions when dealing with extremely small probabilities by considering relatively large ones. Suppose that a safety agency is thinking of establishing a criterion under which an area prone to flooding should be evacuated. The probability of flooding is 1%. There are four possible outcomes:

A No evacuation is necessary, and none is performed.

B An evacuation is performed that is unnecessary.

C An evacuation is performed that is necessary.

D No evacuation is performed, and a flood causes a disaster.

Suppose that the agency is indifferent between the sure outcome **B** and the lottery of **A** with probability p and **D** with probability $1-p$, and between the sure outcome **C** and the lottery of **A** with probability q and **D** with probability $1-q$. Suppose also that it prefers **A** to **D** and that $p \in (0,1)$ and $q \in (0,1)$. Assume that the conditions of the expected utility theorem are satisfied.

(a) Construct a utility function of the expected utility form for the agency.

(b) Consider two different policy criteria:

Criterion 1: This criterion will result in an evacuation in 90% of the cases in which flooding will occur and an unnecessary evacuation in 10% of the cases in which no flooding occurs.

Criterion 2: This criterion is more conservative. It will result in an evacuation in 95% of the cases in which flooding will occur and an unnecessary evacuation in 15% of the cases in which no flooding occurs.

First, derive the probability distributions over the four outcomes under these two criteria. Then, by using the utility function from your answer to (a), decide which criterion the agency would prefer.

3. **Demand for Insurance Cover**

A risk averse individual has a vN-M utility function $u(x)$, depending on consumption x. She has initial wealth w but faces the prospect of incurring a loss L with probability π. Insurance against this loss is available at a premium cost of pB, where $1 > p \geq \pi$, and $L \geq B \geq 0$; that is, to take out coverage B, the consumer pays pB no matter what happens, and receives a payment B from the insurance company in the event of the loss occuring.

(a) Show that the first order condition characterising this individual's optimal level of insurance coverage (assuming an interior solution) is:

$$\pi u'(w - L + (1-p)B)(1-p) - (1-\pi)u'(w - pB)p = 0 \tag{1}$$

(b) If $p = \pi$, that is, the insurance is "actuarily fair", what level of insurance cover would *any* strictly risk averse individual take?

(c) If $p > \pi$ will the individual take out full coverage? If so why, if not why not?

(d) Derive and interpret an expression for the change in the optimal coverage induced by a change in initial wealth w, i.e. dB/dw.

Hint: differentiate (1) with respect to w, to obtain

$$\begin{aligned} &\pi u''(w - L + (1-p)B)(1-p)\left(1 + (1-p)\frac{dB}{dw}\right) \\ &= (1-\pi)u''(w - pB)p\left(1 - p\frac{dB}{dw}\right). \end{aligned}$$

Divide the LHS of this equation by $\pi u'(w - L + (1-p)B)(1-p)$ and the RHS by $(1-\pi)u'(w-pB)p$. Why is this allowable? Noting that $-u''(x)/u'(x) = r_A(x,u)$ is the Arrow-Pratt measure of risk aversion at consumption level x, solve for dB/dw. What assumption about the utility function would enable you to sign this expression?

4. **Risk-preference under state-dependent utility**

   An individual can choose between two suburbs in which to live. The homes in the first suburb are small, while in the second they are large. The vN-M utility index for (aggregate) consumption of other services, x, is $8x^{1/2}$ if he chooses to live in the first suburb, and $5x^{2/3}$ if he chooses to live in the second suburb. Housing costs 20 in the first suburb and 56 in the second, so the indirect utility as a function of initial wealth is

$$v_1(w) = 8(w-20)^{1/2} \text{ if live in first suburb}$$
$$v_2(w) = 5(w-56)^{2/3} \text{ if live in second suburb}$$

   (a) Sketch the two utility indexes. Verify that they cross at $\overline{w} = 120$ and explain what this signifies.

   (b) Suppose that before having invested in housing the individual's endowed wealth is 120. Consider the gamble corresponding to the lottery $((181, 56); (0.5, 0.5))$. What is the expected payoff of this gamble? Compute the individual's utility for each outcome and determine whether the individual would willing exchange this gamble for his initial wealth 120.

   (c) Given an intial wealth of 120, indicate geometrically the optimal actuarially fair gamble for this individual. Explain why the individual wants to undertake such a gamble.

   (d) Can this kind of argument explain why some people gamble regularly?