Rice University

Answer Key for Mid-Semester Examination Fall 2005

ECON 501: Advanced Microeconomic Theory

Part A

1. Consider the following function.

$$V(p_1, p_2, w) = \frac{W}{p_1} - \frac{p_2}{W}$$

State the properties an indirect utility must satisfy and show that the function above satisfies all of them. Derive the uncompensated demand.

Homogeneous of degree 0 in prices and wealth: i.e. $V(\alpha p_1, \alpha p_2, \alpha w) = V(p_1, p_2, w)$

$$\frac{\alpha W}{\alpha p_1} - \frac{\alpha p_2}{\alpha W} = \frac{W}{p_1} - \frac{p_2}{W} \qquad \checkmark$$

Increasing in w:

$$\frac{\partial V}{\partial W} = \frac{1}{p_1} + \frac{p_2}{W^2} > 0 \qquad \checkmark$$

Non-increasing in prices

$$\frac{\partial V}{\partial p_1} = -\frac{W}{p_1^2} < 0 \text{ and } \frac{\partial V}{\partial p_2} = -\frac{1}{W} < 0 \qquad \checkmark$$

Continuous in prices and wealth (for $(p, W) \gg 0$)

Quasiconvex in prices and wealth. Because the function is homogeneous of degree 0 in prices and wealth, it is enough to show the function is quasiconvex in normalized prices: i.e.,

 \checkmark

$$\hat{V}(\hat{p}_1, \hat{p}_2) \equiv V(\hat{p}_1, \hat{p}_2, 1) = \frac{1}{\hat{p}_1} - \hat{p}_2$$

is quasiconvex. But notice both $1/\hat{p}_1$ and $-\hat{p}_2$ are convex functions. Since the sum of convex functions is convex, we have that $\hat{V}(\hat{p}_1, \hat{p}_2)$ is convex and thus quasiconvex as required.

Use Roy's identity to compute the uncompensated demands:

$$\begin{aligned} x_1 \left(p_1, p_2, W \right) &= -\frac{\partial V/\partial p_1}{\partial V/\partial W} = -\frac{-W/p_1^2}{(1/p_1 + p_2/W^2)} = \frac{W^3}{p_1 \left(W^2 + p_1 p_2 \right)} \\ x_2 \left(p_1, p_2, W \right) &= -\frac{\partial V/\partial p_2}{\partial V/\partial W} = -\frac{-1/W}{(1/p_1 + p_2/W^2)} = \frac{p_1 W}{(W^2 + p_1 p_2)} \end{aligned}$$

2. Consider the following indirect utility function.

$$V(p_1, p_2, w) = \left(\frac{-p_1}{W}\right) + \left(\frac{-p_2}{W}\right)$$

Derive the **compensated** demand and the substitution matrix. State the properties a substitution matrix must satisfy and show that these properties all hold for the substitution matrix you have computed.

From fundamental identity:

$$V(p_1, p_2, e(p_1, p_2, u)) = u$$

we have

$$u = \frac{-(p_1 + p_2)}{e(p_1, p_2, u)} \Rightarrow e(p_1, p_2, u) = \frac{-(p_1 + p_2)}{u}.$$

And by Shephard's lemma

$$h_1(p_1, p_2, u) = \frac{\partial e}{\partial p_1} = -\frac{1}{u}$$
$$h_2(p_1, p_2, u) = \frac{\partial e}{\partial p_2} = -\frac{1}{u}$$

The substitution matrix is $S = D_{pp}e = D_ph$.

$$S = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right]$$

The substitution matrix arising from maximization of rational preferences must be symmetric and negative semidefinite. Both are trivially true. Negative semidefiniteness holds since for all vectors $z, z^{T}Sz = 0 \le 0$ (obviously!)

3. State and prove the compensated law of demand for a world with L > 1 commodities and draw a diagram illustrating its implication in a two-commodity world.

Compensated law of demand states there is a negative assocation between the change in compensated demand and prices. Formally,

$$(p - p') \cdot (h(p, u) - h(p', u)) \le 0$$

Proof. Since h(p, u) is a cost-minimizing compensated demand vector for p and h(p', u) is a cost-minizing compensated demand for p', we have

$$e(p, u) = p.h(p, u) \le p.h(p', u) \Rightarrow p.(h(p, u) - h(p', u)) \le 0$$

$$e(p', u) = p'.h(p', u) \le p'.h(p, u) \Rightarrow -p'.(h(p, u) - h(p', u)) \le 0$$

Adding the two inequalites yields

$$(p - p') \cdot (h(p, u) - h(p', u)) \le 0.$$

Part B

4. "Necessities: You can't live by bread alone?" Mr Dubois consumes just 'food' and wine. Let y be the quantity of 'food' he consumes; let x_3 be quantity of wine he consumes; and let w denote his wealth. His preferences between 'food' and wine are given by

$$U(y, x_3) = \alpha \ln y + (1 - \alpha) \ln x_3$$

where $\alpha \in (0, 1)$. Mr Dubois can buy wine but he cannot buy 'food' as such. Rather, he has to buy two basic goods, bread and cheese, and use these as inputs to produce 'food'. His sub-utility function for 'food' is given by: $y = \min \{x_1, 2x_2 - 2\}$ where x_1 is the quantity of cheese, and x_2 is the quantity of bread.

(a) Write down the form of Mr Dubois' overall utility function $u(x_1, x_2, x_3)$ representing his preferences for the three basic goods.

$$u(x_1, x_2, x_3) = \alpha \left(\min \left\{ x_1, 2x_2 - 2 \right\} \right) + (1 - \alpha) \ln x_3.$$

(b) Consider Mr Dubois' production of food and his expenditure on food. Show that Mr Dubois' food expenditure function is given by

$$e(p_1, p_2, y) = \left(p_1 + \frac{p_2}{2}\right)y + p_2$$

where p_1 is the price of cheese, and p_2 is the price of bread.

We want to solve the expenditure minimization problem (EMP)

$$\min_{\langle x_1, x_2 \rangle} p_1 x_1 + p_2 x_2 \text{ s.t. } \min\{x_1, 2x_2 - 2\} \ge y$$

This is the EMP for preferences with L-shaped indifference curves. Minimization occurs at a bundle with $x_1 = 2x_2 - 2 = y$, i.e.,

$$h_1(p_1, p_2, y) = y$$
 and $h_1(p_1, p_2, y) = \frac{y}{2} + 1$

Thus

$$e(p_1, p_2, y) = p_1 h_1(p_1, p_2, y) + p_2 h_2(p_1, p_2, y)$$

= $p_1 y + p_2 \left(\frac{y}{2} + 1\right)$
= $\left(p_1 + \frac{p_2}{2}\right) y + p_2.$

(c) State Shephard's lemma. Verify that this expenditure function satisfies Shephard's lemma and all of the properties an expenditure function is required to satisfy.
 Shepard's lemma:

$$h_{\ell}\left(p,u\right) = \frac{\partial e}{\partial p_{\ell}}$$

For the function derived in part b. we have

$$\frac{\partial e}{\partial p_1} = \frac{\partial}{\partial p_1} \left[\left(p_1 + \frac{p_2}{2} \right) y + p_2 \right] \\
= y = h_1 \left(p_1, p_2, y \right) \quad \checkmark \\
\frac{\partial e}{\partial p_2} = \frac{\partial}{\partial p_2} \left[\left(p_1 + \frac{p_2}{2} \right) y + p_2 \right] \\
= \frac{y}{2} + 1 = h_2 \left(p_1, p_2, y \right) \quad \checkmark$$

Further,

$$e(p_1, p_2, y) = \left(p_1 + \frac{p_2}{2}\right)y + p_2$$

is linear in prices and so is concave and homogeneous of degree one in prices.

$$e(\alpha p_1, \alpha p_2, y) = \left(\alpha p_1 + \frac{\alpha p_2}{2}\right)y + \alpha p_2$$

= $\alpha \left[\left(p_1 + \frac{p_2}{2}\right)y + p_2\right] = \alpha e(p_1, p_2, y)$

It is non-decreasing in prices and increasing in y.

$$\begin{array}{rcl} \frac{\partial e}{\partial p_1} &=& y > 0 & \checkmark \\ \frac{\partial e}{\partial p_2} &=& \frac{y}{2} + 1 > 0 & \checkmark \\ \frac{\partial e}{\partial y} &=& p_1 + \frac{p_2}{2} > 0 & \checkmark \end{array}$$

(d) Consider Mr Dubois choosing quantities y of 'food' and x_3 of wine to maximize his overall utility. Write down the budget constraint for this problem.

Budget constraint for choice y units of 'food' and x_3 units of wine is simply $e(n_1, n_2, y) + n_2 x_2 \leq W$

$$e\left(p_1, p_2, y\right) + p_3 x_3 \le W$$

I.e.,

$$\left(p_1 + \frac{p_2}{2}\right)y + p_2 + p_3 x_3 \le W$$

(e) Derive Mr Dubois' **uncompensated** demand for the basic goods: cheese, bread and wine.

Dubois' utility maximization problem (UMP) is given by

$$\max_{\langle y, x_3 \rangle} \alpha \ln y + (1 - \alpha) \ln x_3 \text{ s.t. } \left(p_1 + \frac{p_2}{2} \right) y + p_2 + p_3 x_3 \le W$$

Yields FONC for interior solution (which are also sufficient - c.f. standard UMP with Cobb-Douglas preferences):

$$y : \frac{\alpha}{y} = \lambda \left(p_1 + \frac{p_2}{2} \right)$$
$$x_3 : \frac{1 - \alpha}{x_3} = \lambda p_3$$
$$BC : \left(p_1 + \frac{p_2}{2} \right) y + p_2 + p_3 x_3 = W$$

Dividing the FONC for y by the FONC for x_3 we obtain

$$\frac{\alpha}{1-\alpha} \times \frac{x_3}{y} = \frac{p_1 + p_2/2}{p_3} \Rightarrow \frac{\alpha}{(1-\alpha)} p_3 x_3 = \left(p_1 + \frac{p_2}{2}\right) y$$

Substituting back into the BC yields

$$\left[\frac{\alpha}{(1-\alpha)} + 1\right] p_3 x_3 = W - p_2 \Rightarrow x_3 \left(p_{1,p_2}, W\right) = \frac{(1-\alpha) \left(W - p_2\right)}{p_3}$$

And

$$y(p_1, p_2, W) = \frac{\alpha (W - p_2)}{p_1 + p_2/2}$$

Utilizing the compensated demands derived in part b, we obtain the uncompensated demands for cheese and bread,

$$x_{1}(p_{1}, p_{2}, W) = h_{1}(p_{1}, p_{2}, y(p_{1}, p_{2}, W))$$

$$= y(p_{1}, p_{2}, W)$$

$$= \frac{\alpha (W - p_{2})}{p_{1} + p_{2}/2}$$

$$x_{2}(p_{1}, p_{2}, W) = h_{2}(p_{1}, p_{2}, y(p_{1}, p_{2}, W))$$

$$= \frac{y}{2} + 1$$

$$= \frac{\alpha (W - p_{2})}{2p_{1} + p_{2}} + 1$$

(f) For Cobb-Douglas preferences, we are used to finding that the expenditure shares $p_j x_j/w$ are constant, and hence that consumption of good j does not depend on prices other than p_j . Give an intuition why this is not the case here for any of the three basic goods (cheese, bread and wine). How could we adapt the notion of an expenditure share appropriately for these preferences so that the adapted expenditure share of wine is constant? Interpret your answer.

Mr Dubois consumption set is given by $X = (0, \infty) \times (1, \infty) \times (0, \infty)$. That is, he must consume more than one unit of bread. So we can view the amount $W - p_2$ as the amount of 'discretionary' wealth Mr Dubois has available once he has purchased one unit of bread. Notice that we have

$$p_3 x_3 (p_1, p_2, W) = (1 - \alpha) (W - p_2)$$

which we may interpret as saying that Mr Dubois spends a constant proportion $(1 - \alpha)$ of his discretionary wealth $W - p_2$ on wine.