2.3 Money Outcomes and Risk Aversion

Ref: MWG 6.C

If individual is a subjective expected utility maximizer, then $\succsim$ over acts can be characterized by π, prob. measure on S representing beliefs and preference scaling utility function $u : X \to \mathbb{R}$.

So can identify act $a = [\delta_{x_1}, E_1; \ldots; \delta_{x_n}, E_n]$ with lottery $L = [x_1, p_1; \ldots; x_n, p_n]$ where $p_i = \pi(E_i)$.

Focus on situation where outcomes are amounts of wealth.

An act is now a random variable $\tilde{x} : S \to X$.

Identify act with its *cumulative distribution function (CDF)*

$$F_{\tilde{x}}(x) = \pi(s \in S : \tilde{x}(s) \leq x)$$

prob. realized outcome no greater than x.

Axioms place no restrictions on preference scaling utility function for wealth, but economics does.

1. u is increasing (or $u'(x) > 0$).
2. u is concave (or $u''(x) \leq 0$)
3. $u'''(x) > 0$ (or $u'(.)$ is convex)

1. "more is better" – local non-satiation
2. Risk aversion
3. Decreasing absolute risk aversion.

Definition 2.3.1: An individual is (weakly) risk averse if for any act $\tilde{x}$, the act that yields

$$\mathrm{E}[\tilde{x}] = \int x dF_{\tilde{x}}(x) \left(= \sum_{x \in X} x\pi(s \in S : \tilde{x}(s) = x) \right)$$

with certainty is weakly preferred to $\tilde{x}$.

Proposition 2.3.1: If

$$U(\tilde{x}) = \int u(x) dF_{\tilde{x}}(x) \left(= \sum_{x \in X} u(x)\pi(s \in S : \tilde{x}(s) = x) \right)$$

represents $\succsim$, then $\succsim$ exhibits (weak) risk aversion *if and only if* the preference-scaling utility function u is concave.

Proof.

I. concave $u \Rightarrow$ risk aversion: By Jensen's inequality, if $u(.)$ is concave then

$$\int u(x) dF(x) \leq u\left(\int x dF(x)\right), \text{ for all } F(.)$$

II. risk aversion $\Rightarrow u$ is concave.
(We will show u not concave $\Rightarrow \succsim$ does not exhibit risk aversion.)

Suppose u is not concave. That is, there exists $y, z \in \mathbb{R}_+$ and $\alpha \in (0,1)$ satisfying

$$u(\alpha y + (1-\alpha) z) < \alpha u(y) + (1-\alpha) u(z).$$

But it then follows for the event E with $\pi(E) = \alpha$ and the act $\tilde{x}$, where

$$\tilde{x}(s) = \begin{cases} y & \text{if } s \in E \\ z & \text{if } s \notin E \end{cases}, \text{ \& so } F_{\tilde{x}}(x) = \begin{cases} 0 & \text{if } x < y \\ \alpha & \text{if } x \in [y,z) \\ 1 & \text{if } x \geq z \end{cases},$$

we have

$$\begin{aligned} \int u(x) dF_{\tilde{x}}(x) &= \alpha u(y) + (1-\alpha) u(z) > u(\alpha y + (1-\alpha) z) \\ &= u\left(\int x dF_{\tilde{x}}(x)\right) \end{aligned}$$

2.4 Measures of Risk Aversion

Certainty Equivalent defined as

$$c(\tilde{x}, u) = u^{-1}\left(\int u(x)\, dF_{\tilde{x}}(x)\right)$$

Obs: If risk averse, then *risk premium* given by

$$\int x dF_{\tilde{x}}(x) - c(\tilde{x}, u)$$

is non-negative.

Probability Risk Premium Consider gamble $\pm\varepsilon$ at base wealth x. *Probability risk premium* implicitly defined by

$$\left(\frac{1}{2} + \pi(x, \varepsilon, u)\right) u(x+\varepsilon) + \left(\frac{1}{2} - \pi(x, \varepsilon, u)\right) u(x-\varepsilon) = u(x)$$

Obs: If risk averse, then $\frac{1}{2} > \pi(x, \varepsilon, u) \geq 0$.

Portfolio Problem: w – initial wealth; $\tilde{x}$ – random return on risky asset with CDF $F_{\tilde{x}}(.)$ and r – riskless return on safe asset.

$$\max_{\alpha \in [0,1]} \int u(\alpha w z + (1-\alpha) w r)\, dF_{\tilde{x}}(z)$$

Solution $\alpha(\tilde{x}, r, w, u)$.

Proposition 2.4.1 Three measures are equivalent in the sense that for two preference-scaling utility functions $v(.)$ and $u(.)$, the following are equivalent.

1. $c(\tilde{x}, v) \leq c(\tilde{x}, u)$ for all $\tilde{x}$
2. $\pi(x, \varepsilon, v) \geq \pi(x, \varepsilon, u)$ for all x, ε
3. $\alpha(\tilde{x}, r, w, v) \leq \alpha(\tilde{x}, r, w, u)$ for all $\tilde{x}, r, w$
4. v is an *increasing* and *concave* transformation of u,
   that is, for some concave function ϕ,

$$v(x) = \phi(u(x))$$

Proof of $(4) \Rightarrow (3)$.

Set $\alpha^* := \alpha(\tilde{x}, r, w, u)$ and set $x^*(z) := \alpha^* wz + (1-\alpha^*) wr$. α^* satisfies the FONC of portfolio problem for u. That is,

$$\int u'(x^*(z))\, w(z-r)\, dF_{\tilde{x}}(z) = 0$$

That is,

$$-\int_{z<r} u'(x^*(z))\, w(r-z)\, dF_{\tilde{x}}(z) + \int_{z>y} u'(x^*(z))\, w(z-r)\, dF_{\tilde{x}}(z) = 0$$

Take the derivative of

$$\int v(w(\alpha wz + (1-\alpha) wr))\, dF(z) \equiv \int \phi(u(w(\alpha wz + (1-\alpha) wr)))\, dF(z)$$

wrt α and evaluate it at α^*.

This yields

$$\begin{aligned} & \int \phi'(u(x^*(z)))\, u'(x^*(z))\, w(z-r)\, dF_{\tilde{x}}(z) \\ = & -\int_{z<r} \phi'(u(x^*(z)))\, u'(x^*(z))\, w(r-z)\, dF_{\tilde{x}}(z) \\ & + \int_{z>r} \phi'(u(x^*(z)))\, u'(x^*(z))\, w(z-r)\, dF_{\tilde{x}}(z) \end{aligned}$$

As $\phi'' \leq 0$,

$$\begin{aligned} \phi'(u(x^*(z))) &\geq \phi'(u(wr)) \text{ for } z<r \\ \phi'(u(x^*(z))) &\leq \phi'(u(wr)) \text{ for } z>r \end{aligned}$$

Hence

$$\begin{aligned} & \int \phi'(u(x^*(z)))\, u'(x^*(z))\, w(z-r)\, dF_{\tilde{x}}(z) \\ \leq & \int \phi'(u(wr))\, u'(x^*(z))\, w(z-r)\, dF_{\tilde{x}}(z) = 0 \end{aligned}$$

So

$$\alpha(\tilde{x}, r, w, v) \leq \alpha^* = \alpha(\tilde{x}, r, w, u)$$

2.4.1 Coefficient of Absolute Risk Aversion

$$r_A(x,u) \equiv -\frac{u''(x)}{u'(x)} \geq 0 \text{ if } u \text{ concave}$$

Proposition 2.4.2

$$v(x) = \phi(u(x)),\ \phi' > 0,\ \phi'' \leq 0 \Leftrightarrow -\frac{v''(x)}{v'(x)} \geq -\frac{u''(x)}{u'(x)} \text{ for all } x.$$

Interpretation. a) $-u''(x)$ is proportional to second-order loss arising from a small fair gamble.

$$\begin{aligned} & \frac{1}{2}u(x+\varepsilon) + \frac{1}{2}u(x-\varepsilon) \\ \approx\ & \frac{1}{2}u(x) + \frac{1}{2}u'(x)\varepsilon + \frac{1}{4}u''(x)\varepsilon^2 + \frac{1}{2}u(x) - \frac{1}{2}u'(x)\varepsilon + \frac{1}{4}u''(x)\varepsilon^2 \end{aligned}$$

Hence

$$u(x) - \frac{1}{2}u(x+\varepsilon) + \frac{1}{2}u(x-\varepsilon) \approx -u''(x)\frac{1}{2}\varepsilon^2$$

b) $u'(x)$ is proportional to first-order loss arising from paying a 'premium' d

$$u(x-d) \approx u(x) - u'(x)d$$

I.e.

$$u(x) - u(x-d) \approx u'(x)d$$

So $-u''(x)/u'(x)$ is proportional to the *marginal rate of substitution* between accepting a small gamble and paying a small premium.

2.4.2 Hypothesis of Decreasing Absolute Risk Aversion

DARA: $r_A(x,u)$ is a decreasing function of x.

Proposition 2.4.3 The following properties are equivalent:

1. The Bernoulli utility index exhibits DARA.
2. Whenever $y < z$, the function $u_y(x) \equiv u(y+x)$ is a *concave* transformation of the function $u_z(x) \equiv u(z+x)$.
3. For any distribution F
$$\int (y+x)\,dF(x) - u^{-1}\left(\int (y+x)\,dF(x)\right)$$
is decreasing in y. That is, the higher is y, the less willing is the individual in paying a premium to get rid of the risk.

4. The probability premium $\pi(x,\varepsilon,u)$ is decreasing in x.
5. For any F, if
$$\int u(x+\varepsilon)\,dF(\varepsilon) \geq u(x)$$
and $x < y$, then
$$\int u(y+\varepsilon)\,dF(\varepsilon) \geq u(y).$$
That is, the set of acceptable risks is *increasing* in x.

Obs. $$\frac{d}{dx}(r_A(x,u)) = \frac{d}{dx}\left(-\frac{u''(x)}{u'(x)}\right) = \frac{-u'''(x)\,u'(x) + (u''(x))^2}{(u'(x))^2}$$

So *necessary* condition for DARA is $u'''(x) > 0$.

Constant absolute risk aversion $u(x) = -\exp(-\alpha x)$.

2.4.3 Hypothesis of Increasing Relative Risk Aversion

Recall portfolio problem and solution $\alpha(F, r, w, u)$.

IRRA says $\alpha(F, r, w, u)$ is *decreasing* in w.

$$r_R(x, u) = -\frac{u''(x)\,x}{u'(x)} \text{ is increasing in } x$$

Constant Relative Risk Aversion:

$$u(x) = \ln x \to r_R(x, u) = 1$$

$$u(x) = \frac{1}{1-\alpha} x^{1-\alpha},\ \alpha \neq 1$$

2.5 Mean-Variance Analysis

Justified in EU Theory if either:

1. Utility function is quadratic

$$u(x) = \begin{cases} x - bx^2/2 & \text{if} \quad x \in [0, 1/b) \\ 1/(2b) & \text{if} \quad x \geq 1/b \end{cases}$$

   or;

2. distributions completely characterized by mean and variance.
   - e.g. class of distributions are all *normally* distributed.

Problems

1. DARA not satisfied.

2. Not all distributions normal, nor characterized just by first two moments.

Consider

$$\tilde{x} = \begin{cases} 100 & \text{if } s \in A \\ 1 & \text{if } s \notin A \end{cases} \text{ with } \pi(A) = 0.2$$

$$\tilde{y} = \begin{cases} 1090 & \text{if } s \in B \\ 10 & \text{if } s \notin B \end{cases} \text{ with } \pi(B) = 0.01$$

$$\mathrm{E}[\tilde{x}] = 0.2 \times 100 + 0.8 \times 1 = 20.8$$

$$\mathrm{E}[\tilde{y}] = 0.01 \times 1090 + 0.99 \times 10 = 20.8$$

$$\mathrm{Var}(\tilde{x}) = (20.8 - 1)^2 \times 0.8 + (100 - 20.8)^2 \times 0.2 = 1,568.16$$

$$\mathrm{Var}(\tilde{y}) = (20.8 - 10)^2 \times 0.99 + (1090 - 20.8)^2 \times 0.01 = 11,547.36$$

Say $u(x) = \ln x$

$$U(\tilde{x}) = 0.8 \ln 1 + 0.2 \ln 100 = 0.92$$

$$U(\tilde{y}) = 0.99 \ln 10 + 0.2 \ln 1090 = 2.35$$

2.6 Stochastic Dominance Relations.

DEFN: The distribution $F(.)$ *first-order stochastically dominates* $G(.)$ if $F(x) \leq G(x)$ for every x.

PROPOSITION: If $F(.)$ second-order stochastically dominates $G(.)$ then

$$\int u(x)\, dF(x) \geq \int u(x)\, dF(x)$$

for every non-decreasing u.

DEFN: The simple distribution $F(.)$ constitutes *an elementary first-order improvement in risk* over the simple distribution $G(.)$ if for some simple distribution $H(.)$, some $\alpha \in [0,1]$, and pair of outcomes $x \geq y$:

$$\begin{aligned} F(.) &= (1-\alpha) H(.) + \alpha\delta_x(.) \\ G(.) &= (1-\alpha) H(.) + \alpha\delta_y(.) \end{aligned}$$

FACT: If $F(.)$ first-order stochastically dominates $G(.)$ then there exists two sequences of simple distributions, $\langle F_n(.)\rangle$ and $\langle G_n(.)\rangle$ such that

$$F_n \to F,\ G_n \to G$$

and for each n, there F_n can be obtained from G_n by a finite sequence of elementary first-order improvements in risk.

DEFN: The distribution $F(.)$ *second-order stochastically dominates* $G(.)$ if $\int_0^x F(t)\,dt \leq \int_0^x G(t)\,dt.$

DEFN: The simple distribution $F(.)$ constitutes *an elementary second-order improvement in risk* over the simple distribution $G(.)$ if for some distribution $H(.)$, some $\alpha, \beta \in [0,1]$, and for some three outcomes $x \geq \beta y + (1-\beta) z$:

$$\begin{aligned} F(.) &= (1-\alpha) H(.) + \alpha \delta_x(.) \\ G(.) &= (1-\alpha) H(.) + \alpha [\beta \delta_y(.) + (1-\beta) \delta_z(.)] \end{aligned}$$

FACT: If $F(.)$ second-order stochastically dominates $G(.)$ then there exists two sequences of simple distributions, $\langle F_n(.)\rangle$ and $\langle G_n(.)\rangle$ such that

$$F_n \to F,\ G_n \to G$$

and for each n, there F_n can be obtained from G_n by a finite sequence of elementary second-order improvements in risk.

PROPOSITION: If $F(.)$ second-order stochastically dominates $G(.)$ then

$$\int u(x)\,dF(x) \geq \int u(x)\,dF(x)$$

for every non-decreasing and *concave* u.