1.8 Aggregation

1.8.1 Aggregation Across Goods

Ref: DM Chapter 5

Motivation:

1. data at group level: food, housing entertainment
   e.g. household surveys

   Q. Can we model this as an ordinary consumer problem over composite commodity groups?

   i.e. $\max U$ defined on composite commodities s.t. budget constraint

2. data on one industry only e.g. I.O. car industry

   - want D among cars to depend on price of food via income effect, but *NOT* to depend on relative prices among different cheeses!

$2'$. Extreme example of 2. *LABOR SUPPLY*

- typically want labor supply to depend on the *real wage* so that it takes account of prices of "all other goods" (one composite good).
- typically want to ignore relative prices of other goods.

N.B. problem $1. \neq 2.$

- our concern here is intuition about problems, not theorems about when no problems.

Do $2'$. then 2. then 1.

Approach 1: Restrict Price Movements

Hicks Composite Commodity Theorem

Suppose prices within the group move *proportionately*

e.g. 3 goods, group commodities 2 & 3.

– like to treat this as if there were 2 goods x_1 & X.

- since prices move proportionately, write

$$p_2 = qp_2^0 \text{ \& } p_3 = qp_3^0 \text{ (i.e. } p_2^0, p_3^0 \text{ `base' prices)}$$

For any (x_2, x_3) let

$$X := p_2^0 x_2 + p_3^0 x_3 \text{ (i.e. qty index using } p^0 \text{ wgts)}$$

Let q be composite price.

Original pblm:-

$$\max u(x_1, x_2, x_3) \text{ s.t. } p_1 x_1 + \underbrace{p_2 x_2 + p_3 x_3}_{qX} \leq w \tag{1}$$

under our assumptions becomes:

$$\max \tilde{u}(x_1, X) \text{ s.t. } p_1 x_1 + qX \leq w \tag{2}$$

Need to check we get same demands. To show this let

$$V\left(p_1, qp_2^0, qp_3^0, w\right) \text{ be indirect utility fn of } (1)$$
$$\tilde{V}(p_1, q, w) \text{ be indirect utility fn of } (2)$$

Notice

$$\tilde{V}(p_1, q, w) \equiv V\left(p_1, qp_2^0, qp_3^0, w\right)$$

Need to show

$$\frac{\partial \tilde{V}}{\partial q} = -\lambda X$$

Check

$$\begin{aligned}\frac{\partial \tilde{V}}{\partial q} &= \frac{\partial V}{\partial p_2} \times \frac{\partial p_2}{\partial q} + \frac{\partial V}{\partial p_3} \times \frac{\partial p_3}{\partial q} \\ &= -\lambda x_2 \times p_2^0 - \lambda x_3 \times p_3^0 = -\lambda X \qquad \checkmark\end{aligned}$$

But needed to assume prices within the group moved together.
I.e. $\Delta p \propto p$.

c.f. when price index works well as approx to "true" index –
"few within group substitutions".

Now consider demand for good 1

$$x_1 = x_1(p_1, q, w)$$

since demand is homogeneous of degree 0 :

$$x_1 = \hat{x}_1\left(\frac{p_1}{q}, \frac{w}{q}\right)$$

e.g. x_1 is leisure, p_1 is nominal wage W, q is price index P (say CPI)

$$\frac{p_1}{q} = \frac{W}{P} \text{ is } \textit{real wage}$$

Q. When will pblms arise with this?

A. When relative prices move and big substitution effects

e.g. communitng cost $\uparrow$ and vacation cost $\downarrow$.

Approach 2: Restrict Preferences

Q. When is D within group of goods *independent* of prices outside the group (except for possibly income effects) [The I.O. Q]

For group g : for each $i \in g$, we have $x_i = x_i(p, w)$ but what we want is

$$x_i = x_i(p_g, w_g) \qquad (*)$$

Intuitively – want a two-step budgeting procedure, where in the first stage we allocate groups' expenditures

$$(w_{g_1}, \ldots, w_{g_N}) \text{ s.t. } \sum_{a=1}^{N} w_{g_a} = w$$

Necessary & sufficient condition for $(*)$ to hold is for $\succsim$ to admit a *weakly separable* representation.

Definition 1. *u is weakly separable, if it can be written as:*

$$u(x) = U\left(u_1(x_{g_1}), \ldots, u_N(x_{g_N})\right), \textit{ where } \frac{\partial U}{\partial u_g} > 0, \textit{ for all } g$$

Example:

$$u(x_1, x_2, x_3) = \ln x_1 + \ln\left(x_2^{1/2} + x_3\right)$$

Proof of Sufficiency (for 2 groups g and h)

Consider overall maximization problem

$$\max_{\langle x \in X \rangle} u(x) \text{ s.t. } p.x \leq w$$

Let x^* be a solution to this problem, that is,

$$V(p,w) = u(x^*) = U\left(u_g\left(x_g^*\right), u_h\left(x_h^*\right)\right)$$

Let

$$w_g^* = \sum_{i \in g} p_i x_i \text{ – expenditure on commodities in group } g$$

We want x_g^* to solve 2^{nd}– stage pblm

$$\max_{x_g} u_g(x_g) \text{ s.t. } \sum_{i \in g} p_i x_i \leq w_g^*$$

I.e. we want sub-pblm to yield *same* condition.

So suppose it does not, that is, $\exists \hat{x}_g$ s.t. $u_g(\hat{x}_g) > u_g\left(x_g^*\right)$ *and* $\sum_{i \in g} p_i \hat{x}_i \leq w_g^*$

But then replace $g-$ component of x^* by $\hat{x}_g \to (\hat{x}_g, x_h^*)$
(This is affordable at (p, w))

This yields

$$\begin{aligned} U\left(u_g(\hat{x}_g), u_h\left(x_h^*\right)\right) &> U\left(u_g\left(x_g^*\right), u_h\left(x_h^*\right)\right) \\ &= u(x^*) = V(p,w) \text{ a contradiction!} \end{aligned}$$

<u>Conclude</u> weak separability is enough for IO problem

- focus on within group allocation
- prices outside the group only affect w_g
  (expenditure on commodities within group).

N.B. Did *not* assume anything about shape of within group preference u_g nor anything about shape of U.

To fix ideas think of following:

Structural example: consumption technology

Think of x_i as 'input' into consumption 'outputs' from which consumer derives final utility.

Say g is 'transport': inputs include car, bike, running shoes

Say h is 'food': inputs include caviar, peanut butter, cabbage, et cetera.

$u_g(.)$ is 'prodn fn' from inputs in to transport utility

- does not depend on caviar or peanut butter consumption.

$u_h(.)$ is 'prodn fn' from inputs to food utility

- does not depend on forms of transport.

$U(.,.)$ represents preferences over 'transport utility' and 'food utility'

What is good about this structural model?

- new goods included without changing 'fundamental' preference model $U(.,.)$
- inputs could household labor

Failures of separability?

Suppose we have *weak separability* (and we have $(u_g)_{g=1}^N$ fns)
Let's use each sub-utility u_{g_a} as a composite commodity: i.e. set $X_g := u_g$.

First-stage of two-stage budgeting problem becomes:

$$\max U(X_1, \ldots, X_N) \text{ s.t. } \sum_{g=1}^{N} e_g(p_g, X_g) \leq w$$

where recall $e_g(p_g, X_g)$ is 2^{nd} stage expenditure at prices p_g to achieve utility level X_g

ALAS, this is not a standard linear budget constraint,
so cannot apply standard theory.

Why? Because $\sum_{a=1}^{N} e_g(p_g, X_g)$ is not linear in X_g.

But if subutility fn u_g is homogeneous then

$$e_g(p_g, X_g) = X_g b_g(p_g)$$

and constraint becomes

$$\sum_{g=1}^{N} b_g(p_g) X_g \leq w$$

BUT RESTRICTIVE.

Can we do better?

Instead of using $X_g = u_g$ use a *money-metric* representation of u_g.
Set

$$X_g := e_g\left(p_g^0, u_g\right)$$

I.e., *expenditure* on group g to achieve 'sub-utility' u_g at base prices p_g^0.

Now,

$$U\left(\ldots,u_g,\ldots\right) \to U(\ldots,V_g(p_g,\underbrace{e_g\left(p_g^0,u_g\right)}_{X_g}),\ldots)$$

with budget constraint

$$\begin{aligned}
&\sum_{g=1}^{N} e_g\left(p_g,u_g\right) \leq w \\
\Leftrightarrow \quad &\sum_{g=1}^{N} e_g\left(p_g^0,u_g\right) \times \frac{e_g\left(p_g,u_g\right)}{e_g\left(p_g^0,u_g\right)} \leq w \\
\Leftrightarrow \quad &\sum_{g=1}^{N} X_g \times P_g \leq w
\end{aligned}$$

where P_g is 'true' price index within g using base utility u_g.

Note:

1. If $u_g\left(.\right)$ homothetic then P_g *independent* of u_g base.

2. Can approximate P_g using Paasche or Lasperyes price indices, provided substitution effects within groups are small. e.g. group prices move together.

1.8.2 Aggregation Across Individuals

Refs: MWG Chapter 4, Varian pp152-154 (Gorman example),
Kreps pp62-63 - (example of agg. D that fails to satisfy WARP)

Address 3 questions about aggregate demand

1. When can aggregate demand be expressed as a function of prices and aggregate wealth?
2. When does aggregate demand satisfy
   - WARP (i.e. substitution matrix for aggregate demand is NSD)?
   - SARP (i.e. substitution matrix for aggregate demand is NSD and symmetric)?
3. When does aggregate demand have welfare significance?

1.8.2.1 Agg Demand and Agg Wealth

Aggregate D $\quad x\left(p, w^1, \ldots, w^I\right) = \sum_i x^i\left(p, w^i\right)$

Q. When can the following hold?

$$\sum_i w^i = \sum_i \widehat{w}^i \Rightarrow \sum_i x^i\left(p, w^i\right) = \sum_i x^i\left(p, \widehat{w}^i\right)$$

Require:

$$\sum_i \frac{\partial}{\partial w^i} x_\ell^i\left(p, w^i\right) dw^i = 0 \text{ for every } \ell$$

$$\text{and every } \left(dw^1, \ldots, dw^I\right) \text{ satisfying } \sum_i dw^i = 0$$

I.e., require

$$\frac{\partial}{\partial w^i} x_\ell^i\left(p, w^i\right) = \frac{\partial}{\partial w^j} x_\ell^j\left(p, w^j\right) \text{ for all } i \text{, all } j \text{ and all } \left(w^1, \ldots, w^I\right)$$

Geometrically: require all wealth expansion paths of all consumers to be *linear* and *parallel*.

Examples:

1. all consumers have identical homothetic preferences

2. all consumers have preferences that are quasi-linear with respect to the same good.

Gorman Form.

A necessary and sufficient condition for a set of consumers to exhibit linear and parallel wealth expansion paths is that the indirect utility functions of the consumers take the following form:

$$v^i\left(p, w^i\right) = a^i(p) + b(p)\,w^i$$

1.8.2.2 WARP and SARP

Notice aggregate demand

$$x\left(p, w^1, \ldots, w^I\right) = \sum_{i=1}^{I} x^i\left(p, w^i\right)$$

satisfies:

1. continuity
2. homogeneity of degree 0 in prices and wealth
3. Walras' Law

$$p.x\left(p, w^1, \ldots, w^I\right) = \sum_{i=1}^{I} w^i$$

Recall WARP

$$\begin{aligned}
\text{If } p.x(\widehat{p}, \widehat{w}) &\leq w \text{ then } \widehat{p}.x(p, w) \geq \widehat{w} \\
\text{If } p.x(\widehat{p}, \widehat{w}) &< w \text{ then } \widehat{p}.x(p, w) > \widehat{w}
\end{aligned}$$

Example from Kreps (pp62-3)

At $p = (10, 10)$, $w^1 = w^2 = 1000$

$$x^1 = (25, 75) \text{ and } x^2 = (75, 25)\text{, so } x = x^1 + x^2 = (100, 100)$$

At $\widehat{p} = (15, 5)$, $\widehat{w}^1 = \widehat{w}^2 = 1000$

$$\widehat{x}^1 = (40, 80) \text{ and } \widehat{x}^2 = (64, 8)\text{, so } \widehat{x} = \widehat{x}^1 + \widehat{x}^2 = (104, 88)$$

Notice

$$\widehat{p}.x^1 = 750 < \widehat{w}^1 \text{ and } p.\widehat{x}^1 = 1200 > w^1,$$

so WARP holds for 1

$$\widehat{p}.x^2 = 1270 > \widehat{w}^2 \text{ and } p.\widehat{x}^2 = 720 < w^2,$$

so WARP holds for 2

But

$$\widehat{p}.x = 2000 = \widehat{w}^1 + \widehat{w}^2 \text{ and } p.\widehat{x} = 1920 < w^1 + w^2,$$

i.e. WARP fails to hold for aggregate demand.

Uncompensated Law of Demand (ULD)

$$(\widehat{p} - p) . \left(x^i \left(\widehat{p}, w^i\right) - x^i \left(p, w^i\right)\right) \leq 0$$

Proposition 1.8.2.1 If for every individual i, $x^i (p, w)$ satisfies ULD then so does aggregate demand

$$x (p, w) = \sum_{i=1}^{I} x^i \left(p, \alpha^i w\right)$$

where α^i is i's share of aggregate wealth. As a consequence, $x (p, w)$ satisfies WARP.

Proof:

(a) $x(p,w)$ satisfies ULD.

To see this, notice that for any two price vectors $\widehat{p}$ and p

$$\begin{aligned} & (\widehat{p}-p).(x(\widehat{p},w)-x(p,w)) \\ = \;& (\widehat{p}-p).\left(\sum_{i=1}^{I} x^i(\widehat{p},\alpha^i w)-\sum_{i=1}^{I} x^i(p,\alpha^i w)\right) \\ = \;& \sum_{i=1}^{I}(\widehat{p}-p)\left(x^i(\widehat{p},\alpha^i w)-x^i(p,\alpha^i w)\right)\leq 0 \end{aligned}$$

(b) $x(p,w)$ satisfies WARP. To verify WARP, assume there exists a pair of price vectors $\widehat{p}$ and p and an aggregate wealth level w, for which

$$p.x(\widehat{p},w)\leq w \text{ and } \widehat{p}.x(p,w)<w,$$

that is, WARP fails to hold. Since by Walras' Law $w=\widehat{p}.x(\widehat{p},w)=p.x(p,w)$, the two inequalities may be reexpressed as

$$\begin{aligned} -p.(x(\widehat{p},w)-x(p,w)) &\geq 0 \\ \text{and } \widehat{p}.(x(\widehat{p},w)-x(p,w)) &> 0 \end{aligned}$$

Adding these we get

$$(\widehat{p}-p).(x(\widehat{p},w)-x(p,w))>0$$

a violation of ULD.

Proposition 1.8.2.2 If each individual's preference relation $\succsim^i$ is homothetic, then $x^i\left(p, w^i\right)$ satisfies ULD.

Proof. Recall

$$\underset{L \times L}{S^i} = \underset{L \times L}{D_p x^i\left(p, w^i\right)} + \underset{L \times 1}{D_w x^i(p, w)} \; \underset{1 \times L}{x^i\left(p, w^i\right)^T}$$

For homothetic preferences, notice

$$D_w x^i(p, w) = \frac{1}{w^i} x^i\left(p, w^i\right)$$

Since

$$\frac{\partial}{\partial w^i} x_\ell^i\left(p, w^i\right) \times \frac{w^i}{x_\ell^i\left(p, w^i\right)} = 1$$

So

$$D_p x^i\left(p, w^i\right) = S^i - \frac{1}{w^i} x^i(p, w)\, x^i(p, w)^T$$

Hence for any vector $z \neq 0$, we have

$$z^T D_p x^i\left(p, w^i\right) z = z^T S^i z - \frac{1}{w^i}\left(z.x^i\left(p, w^i\right)\right)^2 \leq 0$$

Wealth effects are well-behaved.

More generally, for ULD to hold, substitution effects must be large enough to overcome possible 'perversities' arising from wealth effects.

What about SARP?

Require

$$S = D_p x(p, w) + D_w x(p, w)\, x(p, w)^T$$

to be negative semi-definite *and symmetric*.

Implies existence of *positive representative consumer*, $\succsim$, that generates aggregate demand function $x(p, w)$.

1.8.2.3 Aggregate D with Welfare Significance

N.B. Existence of positive representative consumer is *necessary* but *not sufficient* to be able to assign welfare significance to aggregate demand.

Social Welfare: A *Bergson-Samuelson social welfare function (SWF)* is a function

$$W : \mathbb{R}^I \to \mathbb{R}$$

that assigns a social welfare value to each possible vector $\left(u^1, \ldots, u^I\right)$ of utility levels of the I consumers in the economy.
E.g. $W(u_1, \ldots, u_I) = \sum_i u^i$.

Suppose there is a process, that, for any given price vector p, and aggregate wealth level w, redistributes wealth in order to maximise social welfare.

Proposition 1.8.2.3 The value function

$$v(p,w) \quad = \quad \max_{\left\langle w^1,\ldots,w^I\right\rangle} W\left(v^1\left(p,w^1\right),\ldots,v^I\left(p,w^I\right)\right)$$

$$\text{s.t. } \sum_{i=1}^{I} w^i \leq w$$

(where $v^i(p,w)$ is consumer i's indirect utility function and assumed to be concave in wealth) is an indirect utility function of a positive representative consumer for the aggregate demand function

$$x(p,w) = \sum_{i=1}^{I} x^i\left(p, w^i(p,w)\right)$$

Proof:

(a) $v(p,w)$ is an indirect utility function - (exercise)

(b) $x(p,w)$ is demand associated with $v(p,w)$.

To see this notice from the FONC of the wealth distribution problem that (assuming an interior solution):

$$\lambda = \frac{\partial W}{\partial u^i} \times \frac{\partial v^i}{\partial w^i}, \text{ for all } i = 1, \ldots, I \tag{3}$$

From envelope theorem

$$\frac{\partial}{\partial w} v(p,w) = \lambda = \frac{\partial W}{\partial u^i} \times \frac{\partial v^i}{\partial w^i}, \text{ for all } i = 1, \ldots, I$$

For any commodity ℓ:

$$\frac{\partial}{\partial p_\ell} v(p,w) = \sum_{i=1}^{I} \frac{\partial W}{\partial u^i} \times \frac{\partial v^i}{\partial p_\ell} + \lambda \sum_{i=1}^{I} \frac{\partial w^i}{\partial p_\ell} = \sum_{i=1}^{I} \frac{\partial W}{\partial u^i} \times \frac{\partial v^i}{\partial p_\ell}$$

as $\displaystyle\sum_{i=1}^{I} w^i(p,w) = w$ for all (p,w)

So $$x_\ell^R(p,w) = -\frac{\partial}{\partial p_\ell} v(p,w) \Big/ \frac{\partial}{\partial w} v(p,w) = -\sum_{i=1}^{I} \frac{\partial W/\partial u^i \times \partial v^i/\partial p_\ell}{\partial W/\partial u^i \times \partial v^i/\partial w^i}$$

$$= -\sum_{i=1}^{I} \frac{\partial v^i/\partial p_\ell}{\partial v^i/\partial w^i} = \sum_{i=1}^{I} x_\ell^i\left(p, w^i(p,w)\right)$$

Definition: The positive representative consumer $\succsim$ for aggregate demand

$$x(p,w)=\sum_{i=1}^{I}x^{i}\left(p,w^{i}(p,w)\right)$$

is a *normative representative consumer* relative to SWF $W\left(u^1,\ldots,u^I\right)$ if for every (p,w), the distribution $\left(w^1(p,w),\ldots,w^I(p,w)\right)$ solves

$$\max_{\left\langle w^1,\ldots,w^I\right\rangle} W\left(v^1\left(p,w^1\right),\ldots,v^I\left(p,w^I\right)\right)$$

$$\text{s.t. } \sum_{i=1}^{I} w^i \leq w$$

(where $v^i(p,w)$ is consumer i's indirect utility function and assumed to be concave in wealth). Furthermore, the value function $v(p,w)$ of this program is an indirect utility function for $\succsim$.

Example: Suppose $u^i(.)$ are all homogeneous of degree 1, and

$$W\left(u^1,\ldots,u^I\right)=\left(u^1\right)^{\alpha^1}\times\ldots\times\left(u^I\right)^{\alpha^I}\text{, with } \alpha^i>0,\ \sum_{i=1}^{I}\alpha_i=1$$

Optimal wealth distribution function is price-independent rule

$$w^i(p,w)=\alpha^i w$$

Hence in homothetic case, aggregate demand

$$x(p,w)=\sum_{i=1}^{I}x^i\left(p,\alpha^i w\right)$$

may be viewed as originating from the normative representative consumer generated by above SWF.

A very special case: Gorman-form indirect utility

Suppose $v^i\left(p, w^i\right) = a^i(p) + b(p)\, w^i$

1. If $W\left(u^1, \ldots, u^I\right) = \sum_i u^i$ then *any* wealth distribution rule is a welfare maximizing rule for the utilitarian SWF. Hence when indirect utility functions have the Gorman form with common $b(p)$, and SWF is utilitarian, then aggregate demand can *always* be viewed as having been generated by a normative representative consumer.
2. $v(p, w) = \sum_i a^i(p) + b(p)\, w$ is an admissible indirect utility function for the normative representative consumer relative to *any* SWF.

Proof of 2. Enough to show $(\widehat{p}, \widehat{w})$ is socially preferred to (p, w) for $\sum_i u^i$ if and only if $(\widehat{p}, \widehat{w})$ when compared to (p, w) passes the following *potential compensation test*: for any distribution of $\left(w^1, \ldots, w^I\right)$ of w,
there is a distribution of $\left(\widehat{w}^1, \ldots, \widehat{w}^I\right)$ of $\widehat{w}$ such that

$$v^i\left(\widehat{p}, \widehat{w}^i\right) > v^i\left(p, w^i\right) \text{ for all } i.$$

To verify this, suppose

$$\sum_i a^i(\widehat{p}) + b(\widehat{p})\, \widehat{w} - \sum_i a^i(p) - b(p)\, w = c > 0$$

Then $\widehat{w}^i$ that is defined by

$$\begin{aligned} a^i(\widehat{p}) + b(\widehat{p})\, \widehat{w}^i &= a^i(p) + b(p)\, w^i + c/I \\ \text{i.e. } \widehat{w}^i &= \frac{a^i(p) - a^i(\widehat{p}) + b(p)\, w^i + c/I}{b(\widehat{p})} \end{aligned}$$

does the job, since

$$v^i\left(\widehat{p}, \widehat{w}^i\right) = a^i(p) + b(p)\, w^i + c/I > v^i\left(p, w^i\right)$$

$$\begin{aligned} \text{and } \sum_i \widehat{w}^i &= \sum_i \frac{a^i(p) - a^i(\widehat{p}) + b(p)\, w^i + c/I}{b(\widehat{p})} \\ &= \frac{\sum_i \left(a^i(p) - a^i(\widehat{p})\right) + b(p)\, w + c}{b(\widehat{p})} = \widehat{w} \end{aligned}$$