

1.7 Evaluation of Economic Change (Consumer Surplus Analysis)

Ref: MWG 3.I, Varian Chapter 10

Problem At initial situation (p^0, w^0) consumption is $x^0 = x(p^0, w^0)$.
 A change is being considered from $(p^0, w^0) \rightarrow (p^1, w^1)$.
 We wish to evaluate it.

1.7.1 First Order Test

Proposition 1.7.1 If $(p^1 - p^0) \cdot x^0 < w^1 - w^0$ then the consumer is strictly better off under (p^1, w^1) than under (p^0, w^0) .

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Proofs:

1. By Walras's Law, $p^0 \cdot x^0 = w^0$. Hence, $(p^1 - p^0) \cdot x^0 < w^1 - w^0 \Rightarrow p^1 \cdot x^0 < w^1$. I.e., x^0 strictly affordable. Thus by local non-satiation, $\exists \tilde{x}$, s.t. $p^1 \cdot \tilde{x} < w^1$ & $\tilde{x} \succ x^0$. And since $x^1 \succsim \tilde{x}$, we have $x^1 \succ x^0$. ■
2. (Geometry of Indirect Utility Function). Fix 2 vectors, $a, b \in \mathbb{R}^L$.

Cosine of the angle, θ , that *subtends* those 2 vectors is given by

$$\cos \theta = \frac{a \cdot b}{|a| |b|} \text{ and } \cos \theta > 0 \Leftrightarrow 0 < \theta < \pi/2 \text{ (i.e. } 90^\circ)$$

Also notice that from the quasi-convexity of the indirect utility function

$$\text{if } [(p^1, w^1) - (p^0, w^0)] \begin{pmatrix} \frac{\partial}{\partial p_1} v(p^0, w^0) \\ \vdots \\ \frac{\partial}{\partial p_L} v(p^0, w^0) \\ \frac{\partial}{\partial w} v(p^0, w^0) \end{pmatrix} > 0$$

$$\text{then } v(p^1, w^1) > v(p^0, w^0)$$

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Now notice, that

$$\begin{aligned}
 & [(p^1, w^1) - (p^0, w^0)] \begin{pmatrix} \frac{\partial}{\partial p_1} v(p^0, w^0) \\ \vdots \\ \frac{\partial}{\partial p_L} v(p^0, w^0) \\ \frac{\partial}{\partial w} v(p^0, w^0) \end{pmatrix} \\
 &= (p^1 - p^0) \cdot \nabla_p v(p^0, w^0) + (w^1 - w^0) \cdot \frac{\partial}{\partial w} v(p^0, w^0) \\
 &= [-(p^1 - p^0) \cdot x^0 + (w^1 - w^0)] \frac{\partial}{\partial w} v(p^0, w^0) \\
 &> 0 \text{ if and only if } (w^1 - w^0) > (p^1 - p^0) \cdot x^0
 \end{aligned}$$

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Q. What if $(w^1 - w^0) < (p^1 - p^0) \cdot x^0$?

1. Revealed preference: $v(p^1, w^1) ? v(p^0, w^0)$
2. Geometry of indirect utility function.

$$\theta > \pi/2 \Leftrightarrow \cos \theta \left(= \frac{a \cdot b}{|a| |b|} \right) < 0$$

There is an $\bar{\alpha} \in (0, 1)$ such that for $\alpha < \bar{\alpha}$ we have

$$v((1 - \alpha)(p^0, w^0) + \alpha(p^1, w^1)) < v(p^0, w^0)$$

Conclusion: If $|(p^1, w^1) - (p^0, w^0)|$ “small” relative to $|p^1 \cdot x^0 - w^1|$ then first order test is conclusive.

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1.7.2 Using the Expenditure Function

Change could be evaluated by measuring “willingness to pay”.

Money metric indirect utility function (indirect compensation function).

- Fix “base” price vector \bar{p} .
 - $e(\bar{p}, v(p, w))$ is an indirect utility function.
 - $e(\bar{p}, v(p^1, w^1)) - e(\bar{p}, v(p^0, w^0))$ provides a measure of welfare change expressed in \$!
- Two natural candidates for \bar{p} .
 1. initial price vector p^0
 2. new price vector p^1 .

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Let $u^0 = v(p^0, w^0)$ and $u^1 = v(p^1, w^1)$

Equivalent Variation ($\bar{p} = p^0$)

$$\begin{aligned} \text{EV}(p^0, w^0, p^1, w^1) &= e(p^0, u^1) - e(p^0, u^0) \\ &= e(p^0, u^1) - w^0 \end{aligned}$$

“dollar amount that consumer would accept in lieu of change”.

$$\text{i.e. } v(p^0, w^0 + \text{EV}) = u^1$$

Compensating Variation ($\bar{p} = p^1$)

$$\begin{aligned} \text{CV}(p^0, w^0, p^1, w^1) &= e(p^1, u^1) - e(p^1, u^0) \\ &= w^1 - e(p^1, u^0) \end{aligned}$$

“net revenue of planner who must *compensate* consumer after change to bring her back to her original utility”.

$$\begin{aligned} \text{i.e. } v(p^1, w^1 - \text{CV}) &= u^0 \\ \text{EV} > 0 &\Leftrightarrow \text{CV} > 0 \Leftrightarrow x^1 \succ x^0 \end{aligned}$$

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Observations

1. If (p^0, w^0) is being compared with *two* possible changes (p^1, w^1) and (p^2, w^2) then *relative* EV gives indication of which is preferred change.

$$EV^{10} = e(p^0, u^1) - e(p^0, u^0)$$

$$EV^{20} = e(p^0, u^2) - e(p^0, u^0)$$

$$EV^{20} > EV^{10} \Leftrightarrow e(p^0, u^2) > e(p^0, u^1)$$

$$\Leftrightarrow v(p^2, w^2) > v(p^1, w^1)$$

But

$$CV^{10} = e(p^1, u^1) - e(p^1, u^0)$$

$$CV^{20} = e(p^2, u^2) - e(p^2, u^0)$$

$$CV^{20} > CV^{10} \Rightarrow ??$$

2. If \succsim are quasi-linear, i.e. say $u(x) = x_1 + v(x_2, \dots, x_L)$ and $p_1^1 = p_1^0$, then $EV^{10} = CV^{10}$. (Exercise: show this diagrammatically for $L = 2$).

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1.7.3 EV and CV as Areas Below Demand Curves

Assume $p^0 = (p_1^0, \bar{p}_2, \dots, \bar{p}_L)$ and $p^1 = (p_1^1, \bar{p}_2, \dots, \bar{p}_L)$, i.e. only p_1 changes.

We can write: $EV = e(p^0, u^1) - e(p^0, u^0)$

$$= e(p^1, u^1) - e(p^0, u^0) + e(p^0, u^1) - e(p^1, u^1)$$

$$= w^1 - w^0 + e(p^0, u^1) - e(p^1, u^1)$$

$$\text{Similarly } CV = w^1 - w^0 + e(p^0, u^0) - e(p^1, u^0)$$

But since $\frac{\partial}{\partial p_1} e(p_1, \bar{p}_2, \dots, \bar{p}_L, u) = h_1(p_1, \bar{p}_2, \dots, \bar{p}_L, u)$

We have

$$EV = w^1 - w^0 + \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_2, \dots, \bar{p}_L, u^1) dp_1$$

$$CV = w^1 - w^0 + \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_2, \dots, \bar{p}_L, u^0) dp_1$$

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Example: DWL (Excess burden) of specific tax on good 1.

Q.1 How much over and above the tax revenue raised would the consumer be willing to pay to have the tax removed? Or, equivalently, how much worse off is the consumer by raising the revenue via a tax on this commodity as opposed to a lumpsum tax on wealth?

Let $\bar{p}_{-1} = (\bar{p}_2, \dots, \bar{p}_L)$, that is, $p^0 = (p_1^0, \bar{p}_{-1})$ and $p^1 = (p_1^0 + t_1, \bar{p}_{-1})$

Since

$$\begin{aligned} T &= t_1 \times x_1(p^1, w^1) = t_1 \times h_1(p^1, u^1) \\ \text{DWL}^1 &= -T - \text{EV} = e(p^1, u^1) - e(p^0, u^1) - T \\ &= \int_{p_1^0}^{p_1^0 + t_1} h_1(p_1, \bar{p}_{-1}, u^1) dp_1 - t_1 \times h_1(p_1^0 + t_1, \bar{p}_{-1}, u^1) \\ &= \int_{p_1^0}^{p_1^0 + t_1} (h_1(p_1, \bar{p}_{-1}, u^1) - h_1(p_1^0 + t_1, \bar{p}_{-1}, u^1)) dp_1 \end{aligned}$$

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Notice that, $\frac{\partial}{\partial t_1} \text{DWL}^1$

$$= h_1(p_1^0 + t_1, \bar{p}_{-1}, u^1) - h_1(p_1^0 + t_1, \bar{p}_{-1}, u^1) - t_1 \frac{\partial}{\partial p_1} h_1(p_1^0 + t_1, \bar{p}_{-1}, u^1)$$

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Q.2 How much over and above the tax revenue raised must the consumer be paid to leave her as well off as before? Or, equivalently, what would be the deficit a government would run if it compensated the consumer enough to leave her welfare under the tax equal to her pretax welfare?

Now

$$T = t_1 \times h_1(p^1, u^0)$$

So

$$\begin{aligned} DWL^2 &= -T - CV = e(p^1, u^0) - e(p^0, u^0) - T \\ &= \int_{p_1^0}^{p_1^0+t_1} h_1(p_1, \bar{p}_{-1}, u^0) dp_1 - t_1 \times h_1(p_1^0 + t_1, \bar{p}_{-1}, u^0) \\ &= \int_{p_1^0}^{p_1^0+t_1} (h_1(p_1, \bar{p}_{-1}, u^0) - h_1(p_1^0 + t_1, \bar{p}_{-1}, u^0)) dp_1 \end{aligned}$$

Notice that,

$$\frac{\partial}{\partial t_1} DWL^2 = -t_1 \frac{\partial}{\partial p_1} h_1(p_1^0 + t_1, \bar{p}_{-1}, u^0)$$

1.7.4 EV and CV in Several Markets

$$\begin{aligned}
 \text{EV} &= w^1 - w^0 + e(p^0, u^1) - e(p^1, u^1) \\
 &= w^1 - w^0 + \int_{p_1^1}^{p_1^0} h_1(p_1, p_2^1, \dots, p_L^1, u^1) dp_1 \\
 &\quad + \int_{p_2^1}^{p_2^0} h_2(p_1^0, p_2, \dots, p_L^1, u^1) dp_2 \\
 &\quad \vdots \\
 &\quad + \int_{p_\ell^1}^{p_\ell^0} h_\ell(p_1^0, p_2^0, \dots, p_{\ell-1}^0, p_\ell, p_{\ell+1}^1, \dots, p_L^1, u^1) dp_\ell \\
 &\quad \vdots \\
 &\quad + \int_{p_L^1}^{p_L^0} h_L(p_1^0, p_2^0, \dots, p_{L-1}^0, p_L, u^1) dp_L
 \end{aligned}$$

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Example: $L = 3$. Assume goods 1 and 2 are Hicksian substitutes, that is,

$$\frac{\partial}{\partial p_2} h_1(p_1, p_2, p_3, u) = \frac{\partial}{\partial p_1} h_2(p_1, p_2, p_3, u) > 0$$

Consider change,

$$p_1^1 > p_1^0, p_2^1 < p_2^0 \text{ and } p_3^1 = p_3^0 = \bar{p}_3$$

(Refer to diagram in lecture)

$$\text{EV} = \int_{p_1^1}^{p_1^0} h_1(p_1, p_2^1, \bar{p}_3, u^1) dp_1 + \int_{p_2^1}^{p_2^0} h_2(p_1^0, p_2, \bar{p}_3, u^1) dp_2$$

So

$$\text{EV} = -A + B$$

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Now consider path $(p_1^1, p_2^1, \bar{p}_3) \rightarrow (p_1^1, p_2^0, \bar{p}_3) \rightarrow (p_1^0, p_2^0, \bar{p}_3)$

$$EV = \int_{p_2^1}^{p_2^0} h_2(p_1^1, p_2, \bar{p}_3, u^1) dp_2 + \int_{p_1^1}^{p_1^0} h_1(p_1, p_2^0, \bar{p}_3, u^1) dp_1$$

$$EV = B + C - (A + D)$$

Symmetry of substitution matrix $\Rightarrow D = C$. E.g., say $\frac{\partial h_1}{\partial p_2} = \frac{\partial h_2}{\partial p_1} = \text{constant}$

$$D = [p_1^1 - p_1^0] \times \frac{\partial h_1}{\partial p_2} [p_2^0 - p_2^1] \quad \text{and} \quad C = [p_2^0 - p_2^1] \times \frac{\partial h_2}{\partial p_1} [p_1^1 - p_1^0]$$

More generally,

$$\begin{aligned} D &= \int_{p_1^0}^{p_1^1} \left(\int_{p_2^1}^{p_2^0} \frac{\partial}{\partial p_2} h_1(p_1, p_2, \bar{p}_3, u^1) dp_2 \right) dp_1 \\ &= \int_{p_1^0}^{p_1^1} \left(\int_{p_2^1}^{p_2^0} \frac{\partial}{\partial p_1} h_2(p_1, p_2, \bar{p}_3, u^1) dp_2 \right) dp_1 \\ &= \int_{p_2^1}^{p_2^0} \left(\int_{p_1^0}^{p_1^1} \frac{\partial}{\partial p_1} h_2(p_1, p_2, \bar{p}_3, u^1) dp_1 \right) dp_2 = C \end{aligned}$$

1.7.5 Approximations - Consumer Surplus

$$\begin{aligned}
 CS &= w^1 - w^0 \\
 &+ \int_{p_1^1}^{p_1^0} x_1(p_1, p_2^1, \dots, p_L^1, w^1) dp_1 \\
 &+ \int_{p_2^1}^{p_2^0} x_2(p_1^0, p_2, \dots, p_L^1, w^1) dp_2 \\
 &\vdots \\
 &+ \int_{p_\ell^1}^{p_\ell^0} x_\ell(p_1^0, p_2^0, \dots, p_{\ell-1}^0, p_\ell, p_{\ell+1}^1, \dots, p_L^1, w^1) dp_\ell \\
 &\vdots \\
 &+ \int_{p_L^1}^{p_L^0} x_L(p_1^0, p_2^0, \dots, p_{L-1}^0, p_L, w^1) dp_L
 \end{aligned}$$

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1. Result no longer independent of path unless

$$\frac{\partial x_\ell}{\partial p_k} = \frac{\partial x_k}{\partial p_\ell}$$

2. If no income effects for goods whose price change, then CS *is exact*.

- e.g. quasi-linear utility, $u(x) = x_1 + v(x_2, \dots, x_L)$
Marshallian case of *constant* MU of numeraire (take $p_1 = 1$).

3. If income effect small, then procedure approx. correct.

- Marshall argued if good ℓ just one commodity among many, then extra wealth spread around and income effect will be small.

CAVEAT – fallacy of composition – deal with non-negligible fraction of commodities then approx error may not be small in aggregate.

- approx error may be large as fraction of DWL.

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1.7.3 Cost of Living Indices

“TRUE”

1.

$$\frac{e(p^1, u^0)}{e(p^0, u^0)}$$

Analog of CV but a ratio. Recall

$$\begin{aligned} CV &= e(p^1, u^1) - e(p^1, u^0) \\ &= w^1 - w^0 + e(p^0, u^0) - e(p^1, u^0) \end{aligned}$$

2.

$$\frac{e(p^1, u^1)}{e(p^0, u^1)}$$

Analog of EV but a ratio. Recall

$$\begin{aligned} EV &= e(p^0, u^1) - e(p^0, u^0) \\ &= w^1 - w^0 + e(p^0, u^1) - e(p^1, u^1) \end{aligned}$$

Approximations

1. Laspeyres Price Index

$$L = \frac{p^1 \cdot x^0}{p^0 \cdot x^0} \geq \frac{e(p^1, u^0)}{e(p^0, u^0)}$$

2. Paasche Price Index

$$P = \frac{p^1 \cdot x^1}{p^0 \cdot x^1} \leq \frac{e(p^1, u^1)}{e(p^0, u^1)}$$

When are approximations good?

Small substitution effects, so $h(p^0, u^0)$ is 'close' to $h(p^1, u^0)$
 [c.f. CS is good approx. to EV & CV when income effects are small.]

e.g.

- (a) Leontieff preferences
- (b) If Δp proportional to p . Formally,

$$\begin{aligned} e(p^1, u^0) &\approx e(p^0, u^0) + \nabla_p e(p^0, u^0) (p^1 - p^0) \\ &\quad + \frac{1}{2} (p^1 - p^0) D_{pp} e(p^0, u^0) (p^1 - p^0) \\ &= e(p^0, u^0) + (p^1 - p^0) \cdot x^0 + \frac{1}{2} (p^1 - p^0) S (p^1 - p^0) \end{aligned}$$

Divide by $e(p^0, u^0) = p^0 \cdot x^0 \rightarrow$

$$\frac{e(p^1, u^0)}{e(p^0, u^0)} \approx \frac{p^1 \cdot x^0}{p^0 \cdot x^0}$$

Conclusion: Laspeyres index is like a linear approx. of true index at u^0 .
Good for small Δp or for Δp close to proportional to p .

Q. When is

$$\frac{e(p^1, u^0)}{e(p^0, u^0)} = \frac{e(p^1, u^1)}{e(p^0, u^1)}?$$

A. Homothetic preferences: $e(p, u) = ub(p)$. In this case case

$$L \geq \text{'true'} \geq P.$$