1.6 Duality, Integrability & Revealed Preference

(Ref: MWG 3.H, 2.F and 3.J)

3 closely related questions:-

1. Starting with an expenditure or indirect utility function can we work "backwards" to recover the underlying *direct* utility function that would have generated it? (*DUALITY THEORY*)
2. Given a function $x(p, w)$ that is claimed to be an uncompensated demand function, is there some locally insatiable preference-maximizing consumer behind it? (*INTEGRABILITY PROBLEM*)
3. Viewing choice behavior as primitive, when is a set of observed choices of a consumer consistent with our preference-based model of choice? (*REVEALED PREFERENCE*)

1.6.1 Duality Theory

Constructing a utility function from an expenditure function $e(p, u)$.

Define
$$V_u(p) \equiv \left\{x \in \mathbb{R}_+^L \mid p.x \geq e(p, u)\right\}$$

Define
$$V_u \equiv \bigcap_{p \gg 0} V_u(p) \equiv \left\{x \in \mathbb{R}_+^L \mid p.x \geq e(p, u) \text{ for all } p \gg 0\right\}$$

Define
$$\begin{aligned} u^*(x) &\equiv \max_u \{x \in V_u\} \\ &\equiv \max_u \{p.x \geq e(p, u) \text{ for all } p \gg 0\} \end{aligned}$$

Theorem 1: $u^*(\cdot)$ is quasi-concave and continuous in x.

Theorem 2: $e(p, u) = \min_x p.x$ s.t. $u^*(x) \geq u$.

1.6.2 Integrability Problem

Given $x(p, w)$ does there exist a locally insatiable preference-maximizing consumer for whom $x(p, w)$ is the solution to that consumer's UMP.

We know it is necessary $x(p, w)$ satisfy

1. homogeneity $x(\alpha p, \alpha w) = x(p, w)$ for all $\alpha > 0$
2. Walras's Law $p.x(p, w) = w$
3. symmetry and negative semi-definiteness of substitution matrix

$$\begin{aligned} & \left[\frac{\partial}{\partial p_k} x_\ell(p, w) + \frac{\partial}{\partial w} x_\ell(p, w) \times x_k(p, w)\right] \\ = \quad & D_p x(p, w) + D_w x(p, w)\, x(p, w)^T \end{aligned}$$

Punchline: Subject to techn. caveats, (1), (2) & (3) sufficient as well.

Method: Pick some point $x^0 = x\left(p^0, w^0\right)$ and assign it utility u^0.

Shephard's lemma and non-satiation gives us the following system of equations

$$\frac{\partial}{\partial p_\ell} e\left(p, u^0\right) \equiv h_\ell\left(p, u^0\right) \equiv x_\ell\left(p, e\left(p, u^0\right)\right)$$

with initial condition

$$e\left(p^0, u^0\right) = p^0.x\left(p^0, w^0\right) = p^0.x^0 = w^0$$

Fundamental result of theory of partial differential equations is that above system will have solution if and only if

$$\frac{\partial}{\partial p_k} h_\ell\left(p, u^0\right) = \frac{\partial}{\partial p_\ell} h_k\left(p, u^0\right) \text{ for all } \ell \text{ and } k.$$

But this follows from symmetry of Slutsky/substitution matrix.

1.6.3 Revealed Preference

(see also Kreps pp44-5, Varian pp131-9.)

Viewing choice behavior as primitive, when is a set of observed choices of a consumer consistent with our preference-based model of choice?

Data is:-

for	(p^1, w^1)	$\ldots$	(p^N, w^N)
we observe	x^1	$\ldots$	x^N

Assume local non-satiation, i.e. $p^n.x^n = w^n$ for $n = 1, \ldots, N$

Our aim is to construct preferences from revealed choices.

Definition: Take any finite set of demand data.

1. If $p^m.x^n \leq w^m$ then the data are said *to reveal* $x^m \succsim^* x^n$
2. If $p^m.x^n < w^m$ then the data are said *to reveal* $x^m \succ^* x^n$

Strong Axiom of Revealed Preference (SARP)

Data satisfy SARP if, for preferences that the data reveal, you *cannot construct* a cycle of length less than or equal to $N+1$ of the form

$$x^{\pi(1)} \succsim^* x^{\pi(2)} \succsim^* \ldots \succsim^* x^{\pi(1)}$$

where $\pi : \{1, \ldots, N\} \to \{1, \ldots, N\}$ is a permutation, and at least one of the $\succsim^*$s is a $\succ^*$.

Theorem (Houthakker): A finite set of demand data satisfies SARP iff these data are consistent with max. of locally insatiable preferences.

Obs: For 2 goods, following weaker axiom is both necessary & sufficient.

Weak Axiom of Revealed Preference (WARP)

Let x^i be chosen for (p^i, w^i), $i = 1, 2$. Then for $j \neq k$

1. If $p^j.x^k \leq w^j$ then $p^k.x^j \geq w^k$
2. If $p^j.x^k < w^j$ then $p^k.x^j > w^k$

3 Good Example: Satisfies Warp but Violates SARP.

Suppose our consumer lives in a three-commodity world.

- Prices $p^1 = (3, 2, 4)$ & wealth $w = 15$ consumer chooses $x^1 = (1, 2, 2)$
- Prices $p^2 = (4, 3, 2)$ & wealth $w = 15$ consumer chooses $x^2 = (2, 1, 2)$
- Prices $p^3 = (2, 4, 3)$ & wealth $w = 15$ consumer chooses $x^3 = (2, 2, 1)$

		Bundles		
		x^1	x^2	x^3
Prices	p^1	15	16	14
	p^2	14	15	16
	p^3	16	14	15

Table entries are cost of the 3 bundles at each of the 3 sets of prices

1.6.3 Comparative Statics from Revealed Preference

Recall *Hicksian* compensation:-

$$\begin{aligned} &\text{for } p \to p + \triangle p && \text{provide } \triangle w \text{ so that} \\ x_\ell (p + \triangle p, w + \triangle w) &\equiv h_\ell (p + \triangle p, u) && \text{for each } \ell \\ \text{i.e. } w + \triangle w &\equiv e(p + \triangle p, u) \end{aligned}$$

where u is *original* level of utility achieved at (p, w)

Slutsky compensation:-

$$\begin{aligned} &\text{for } p \to p + \triangle p && \text{provide } \triangle w \\ \text{so that } w + \triangle w &\equiv (p + \triangle p) . x (p, w) \end{aligned}$$

I.e. compensation makes *pre-change* consumption bundle affordable.

(Over) Compensated Law of Demand. If Choice behavior obeys WARP then for a(n over-)compensated price change $(\triangle p, \triangle w)$:

$$\triangle p . \triangle x \leq 0$$

Differential Analog: (Assume uncompensated demand is now a continuously differentiable function).

Infinitesimal compensation $dw = x(p, w).dp$

So for compensated change (dp, dw)

$$\begin{aligned} dx &= D_{p,w}x(p, w) \begin{bmatrix} dp \\ dw \end{bmatrix} \\ &= \begin{bmatrix} D_p x(p, w) & D_w x(p, w) \end{bmatrix} \begin{bmatrix} dp \\ dw \end{bmatrix} \\ &= \begin{bmatrix} D_p x(p, w) & D_w x(p, w) \end{bmatrix} \begin{bmatrix} I \\ x(p, w)^T \end{bmatrix} dp \end{aligned}$$

Define $S(p, w)$ as the Slutsky/substitution matrix

$$S(p, w) = D_p x(p, w) + D_w x(p, w)\, x(p, w)^T$$

Then $\triangle p.\triangle x \leq 0$ in the limit becomes

$$dp.dx = dp^T S(p, w)\, dp \leq 0$$

I.e. $S(p, w)$ is *negative semi-definite.*

WARP $\rightarrow$ negative semi-definiteness of substitution matrix
(i.e. [over-]compensated law of demand)

SARP $\rightarrow$ nsd & symmetry of substitution matrix
$\rightarrow$ transitivity of choice!