1.2 Utility Maximization Problem (UMP)

(MWG 2.D, 2.E; Kreps 2.2) $\max_{\langle x \in X \rangle} u(x)$ s.t. $p.x \leq w$ and $x \geq 0$

For a cts preference relation represented by a cts utility fn, $u(\cdot)$:

1. The UMP has at least *one* solution for all strictly positive prices and non-negative levels of income.
2. If x is a solution of the UMP for given p and w, then x is also a solution for (ap, aw) for any positive scalar a.

   i.e. $x(p,w) \equiv x(ap, aw)$ [Homogeneity of degree 0 of demand.]
3. If in addition we assume preferences are locally non-satiated then x being a solution of the UMP implies $\sum_\ell p_\ell x_\ell = w$.
4. If in addition we assume preferences are convex (i.e. u is quasi-concave) then the set of solutions $x(p,w)$ to the UMP is a *convex* set.
5. If preferences are strictly convex then the solution to the UMP is unique and $x(p,w)$ is a continuous function of p and w.

1.3 The Indirect Utility Function

Assume that $u(\cdot)$ is a cts fn that represents locally non-satiated preferences. The indirect utility fn $v(p,w)$ is:-

1. homogeneous of degree zero in p and w

   i.e. $v(p,w) \equiv v(ap, aw)$ for all $a > 0$
2. strictly increasing in w
3. non-increasing in p
4. quasi-convex in p and w, that is

$$v(\alpha p + [1-\alpha] p', \alpha w + [1-\alpha] w') \leq \max[v(p,w), v(p', w')]$$

1.4 The Expenditure Minimization Problem (EMP)

For a cts preference relation represented by a cts utility function, $u(\cdot)$:

1. The EMP has at least *one* solution for all strictly positive prices & $u \geq u(0)$.
2. If x is a solution of the EMP for given p and u, then x is also a solution for (ap, u) for any positive scalar a.

   i.e. $h(p, u) \equiv h(ap, u)$ [Homogeneity of degree 0 in prices.]
3. If in addition we assume preferences are convex (i.e. u is quasi-concave) then the set of solutions $h(p, u)$ to the EMP is a *convex* set.
4. If preferences are strictly convex then the solution to the EMP is unique and $h(p, u)$ is a continuous function of p and u.

Properties of the Expenditure Function

Assume that $u(\cdot)$ is a cts fn that represents locally non-satiated preferences. The expenditure function $e(p, u)$ is:-

1. homogeneous of degree one in p

   i.e. $e(p, u) \equiv ae(p, u)$ for all $a > 0$
2. strictly increasing in u
3. non-decreasing in p
4. concave in p, that is

$$e(\alpha p + [1-\alpha] p', u) \geq \alpha e(p, u) + (1-\alpha) e(p', u)$$

1.5 UMP & EMP with Derivatives

Constrained Optimatization and the Kuhn-Tucker Conditions

(reference: MWG appendix M.K; Kreps Appendix A)

Problem

$$\begin{aligned} &\max_{x \in \mathbb{R}^N} f(x) \\ \text{s.t. } g_m(x) &= 0,\ m = 1, \ldots, M \\ h_k(x) &\leq 0,\ k = 1, \ldots, K \end{aligned}$$

Form the Lagrangian function:-

$$\mathcal{L}(x, \mu, \lambda) = f(x) - \sum_{m=1}^{M} \mu_m g_m(x) - \sum_{k=1}^{K} \lambda_k h_k(x)$$

THEOREM (Assuming the constraint qualification is satisfied) For (x^*, μ^*, λ^*) to be a solution to the above constrained optimization problem, (x^*, μ^*, λ^*) must satisfy

(i) $\frac{\partial}{\partial x_n} \mathcal{L}(x^*, \mu^*, \lambda^*) = 0$ for all $n = 1, \ldots, N$

(ii) $\mathcal{L}(x^*, \mu^*, \lambda^*) = f(x^*)$ and $\lambda_k^* \geq 0$, for all $k = 1, \ldots, K$

(i) and (ii) can be re-expressed as the Kuhn-Tucker FONCs for a maximum:

(A) $\frac{\partial}{\partial x_n} f(x^*) = \sum_{m=1}^{M} \mu_m^* \frac{\partial}{\partial x_n} g_m(x^*) + \sum_{k=1}^{K} \lambda_k^* \frac{\partial}{\partial x_n} h_k(x^*)$, $\forall\ n = 1, \ldots, N$

$$
\begin{aligned}
\text{(B) } \lambda_k^* h_k(x^*) &= 0\ \forall\ k = 1, \ldots, K \text{ \& } g_m(x^*) = 0,\ \forall\ m = 1, \ldots, M. \\
&\quad \text{which implies } \textit{complementary slackness}\text{, i.e.,} \\
\lambda_k^* &> 0 \Rightarrow h_k(x^*) = 0 \text{ and } h_k(x^*) < 0\ \Rightarrow \lambda_k^* = 0
\end{aligned}
$$

For UMP

$$
\begin{aligned}
\max_{x \in \mathbb{R}^L} u(x) \qquad & \text{s.t. } -x_\ell \le 0,\ \ell = 1, \ldots, L\ ;\ \mu_\ell \\
\sum_{\ell=1}^{L} p_\ell x_\ell - w &\le 0,\ k = 1, \ldots, K\ ;\ \lambda
\end{aligned}
$$

Form the Lagrangian fn:

$$
\mathcal{L}(x, \mu, \lambda) = u(x) - \sum_{\ell=1}^{L} \mu_\ell(-x_\ell) - \lambda\left(\sum_{\ell=1}^{L} p_\ell x_\ell - w\right)
$$

K-T FONC

(A) $$\frac{\partial}{\partial x_\ell} u(x^*) = -\mu_\ell^* + \lambda^* p_\ell,$$

(B) $$\mu_\ell^* > 0 \Rightarrow x_\ell^* = 0 \text{ \& } x_\ell^* > 0 \Rightarrow \mu_\ell^* = 0$$

$$\lambda^* > 0 \Rightarrow \sum_{\ell=1}^{L} p_\ell x_\ell^* = w \text{ \& } \sum_{\ell=1}^{L} p_\ell x_\ell^* < w \Rightarrow \lambda^* = 0$$

$$\begin{aligned} v(p,w) &= \mathcal{L}(x^*, \mu^*, \lambda^*) \\ &= u(x^*) + \sum_{\ell=1}^{L} \mu_\ell^* x_\ell^* - \lambda^* \left(\sum_{\ell=1}^{L} p_\ell x_\ell^* - w \right) = u(x^*) \end{aligned}$$

For EMP

$$\max_{x \in \mathbb{R}^L} -\sum_{\ell=1}^{L} p_\ell x_\ell \qquad \text{s.t.} \quad -x_\ell \leq 0,\ \ell = 1, \ldots, L\ ;\ \mu_\ell$$

$$u - u(x) \quad \leq \quad 0,\ k = 1, \ldots, K\ ;\ \gamma$$

Form the Lagrangian fn:

$$\mathcal{Z}(x, \mu, \gamma) = -\sum_{\ell=1}^{L} p_\ell x_\ell - \sum_{\ell=1}^{L} \mu_\ell (-x_\ell) - \gamma (u - u(x))$$

K-T FONC

$$\text{(A)} \qquad p_\ell = \gamma^* \frac{\partial}{\partial x_\ell} u(x^*) + \mu_\ell^*,$$

$$\text{(B)} \qquad \mu_\ell^* > 0 \Rightarrow x_\ell^* = 0 \ \& \ x_\ell^* > 0 \Rightarrow \mu_\ell^* = 0$$

$$\gamma^* > 0 \Rightarrow u(x^*) = u \ \& \ u(x^*) > u \Rightarrow \gamma^* = 0$$

$$\begin{aligned} e(p,u) &= -\mathcal{Z}(x^*, \mu^*, \gamma^*) \\ &= \sum_{\ell=1}^{L} p_\ell x_\ell^* - \sum_{\ell=1}^{L} \mu_\ell^* x_\ell^* + \gamma^*(u - u(x^*)) = \sum_{\ell=1}^{L} p_\ell x_\ell^* \end{aligned}$$

The Envelope Theorem

THEOREM *For the problem*

$$\max_{\langle x \rangle} f(x;q) \quad \text{s.t.} \quad \begin{aligned} & g_m(x;q) = 0, \text{ for } m = 1, \ldots, M \\ & h_k(x;q) \leq 0, \text{ for } k = 1, \ldots, K, \\ & x_n \geq 0, \text{ for } n = 1, \ldots, N \end{aligned}$$

Let

$$\mathcal{L}(x, \mu, \lambda; q) = f(x;q) - \sum_{m=1}^{M} \mu_m g_m(x;q) - \sum_{k=1}^{K} \lambda_k h_k(x;q).$$

And let (x^*, μ^*, λ^*) *be a solution to the K-T FONCs, so that*

$$v(q) = f(x^*, q).$$

The Envelope Theorem

Then

$$\begin{aligned}\frac{dv(q)}{dq} &= \frac{\partial \mathcal{L}(x^*, \mu^*, \lambda^*; q)}{\partial q} \\ &= \frac{\partial f(x^*; q)}{\partial q} - \sum_{m=1}^{M} \mu_m^* \frac{\partial g_m(x^*; q)}{\partial q} - \sum_{k=1}^{K} \lambda_k^* \frac{\partial h_k(x^*; q)}{\partial q}.\end{aligned}$$

Proof of Envelope Theorem

By direct differentiation:-

$$\frac{dv(q)}{dq} = \sum_{n=1}^{N} \frac{\partial f(x^*; q)}{\partial x_n} \frac{dx_n^*}{dq} + \frac{\partial f(x^*; q)}{\partial q}$$

But from K-T FONCs (A)

$$\frac{\partial f(x^*; q)}{\partial x_n} = \sum_{m=1}^{M} \mu_m^* \frac{\partial g_m(x^*; q)}{\partial x_n} + \sum_{k=1}^{K} \lambda_k^* \frac{\partial h_k(x^*; q)}{\partial x_n} \tag{1}$$

unless $x_n \geq 0$ constraint binds in which case $x_n^* = 0$ & $dx_n^*/dq = 0$.

So multiplying (1) by dx_n^*/dq and summing over n leads to:

$$\sum_{n=1}^{N} \frac{\partial f(x^*; q)}{\partial x_n} \frac{dx_n^*}{dq} = \sum_{n=1}^{N} \left[\sum_{m=1}^{M} \mu_m^* \frac{\partial g_m(x^*; q)}{\partial x_n} + \sum_{k=1}^{K} \lambda_k^* \frac{\partial h_k(x^*; q)}{\partial x_n} \right] \frac{dx_n^*}{dq} \tag{2}$$

Now from K-T FONCs (B) we have:

$$\sum_{m=1}^{M} \mu_m^* g_m(x^*;q) + \sum_{k=1}^{K} \lambda_k^* h_k(x^*;q) \equiv 0 \tag{3}$$

Differentiating (3) wrt q:

$$\begin{aligned} & \sum_{m=1}^{M} \frac{\partial \mu_m^*}{\partial q} g_m(x^*;q) + \sum_{m=1}^{M} \mu_m^* \frac{\partial g_m(x^*;q)}{\partial q} + \sum_{n=1}^{N}\sum_{m=1}^{M} \mu_m^* \frac{\partial g_m(x^*;q)}{\partial x_n}\frac{dx_n^*}{dq} \\ & + \sum_{k=1}^{K} \frac{\partial \lambda_k^*}{\partial q} h_k(x^*;q) + \sum_{k=1}^{K} \lambda_k^* \frac{\partial h_k(x^*;q)}{\partial q} + \sum_{n=1}^{N}\sum_{k=1}^{K} \lambda_k^* \frac{\partial h_k(x^*;q)}{\partial x_n}\frac{dx_n^*}{dq} \\ = & \ 0 \end{aligned} \tag{4}$$

The first term of the LHS is zero as $g_m(x^*;q) = 0$, and the fourth term is also zero as recall by the complementary slackness conditions either h_k binds in which case $h_k(x^*;q) = 0$, or it is 'slack' in which case $\lambda_k^* = 0$ and $\partial \lambda_k^* / \partial q = 0$.

Hence from (4)

$$\begin{aligned} & \sum_{m=1}^{M} \mu_m^* \frac{\partial g_m(x^*;q)}{\partial q} + \sum_{k=1}^{K} \lambda_k^* \frac{\partial h_k(x^*;q)}{\partial q} \\ = & \ -\sum_{n=1}^{N}\sum_{m=1}^{M} \mu_m^* \frac{\partial g_m(x^*;q)}{\partial x_n}\frac{dx_n^*}{dq} - \sum_{n=1}^{N}\sum_{k=1}^{K} \lambda_k^* \frac{\partial h_k(x^*;q)}{\partial x_n}\frac{dx_n^*}{dq} \end{aligned} \tag{5}$$

So combining (2) and (5) we obtain:

$$\sum_{n=1}^{N} \frac{\partial f(x^*;q)}{\partial x_n}\frac{dx_n^*}{dq} = -\sum_{m=1}^{M} \mu_m^* \frac{\partial g_m(x^*;q)}{\partial q} - \sum_{k=1}^{K} \lambda_k^* \frac{\partial h_k(x^*;q)}{\partial q}$$

and hence the desired result:

$$\frac{dv(q)}{dq} = \frac{\partial f(x^*;q)}{\partial q} - \sum_{m=1}^{M} \mu_m^* \frac{\partial g_m(x^*;q)}{\partial q} - \sum_{k=1}^{K} \lambda_k^* \frac{\partial h_k(x^*;q)}{\partial q}.$$

Applications of the Envelope Theorem.

(a) Roy's Identity:

$$x_\ell(p,w) = -\frac{\partial v(p,w)/\partial p_\ell}{\partial v(p,w)/\partial w}$$

Proof: By the envelope theorem

$$\begin{aligned}
\frac{\partial v(p,w)}{\partial p_\ell} &= \frac{\partial}{\partial p_\ell}\mathcal{L}(x^*,\mu^*,\lambda^*;p,w) \\
&= \frac{\partial}{\partial p_\ell}\left[u(x^*) + \sum_{\ell=1}^{L}\mu_\ell^* x_\ell^* - \lambda^*\left(\sum_{\ell=1}^{L} p_\ell x_\ell^* - w\right)\right] = -\lambda^* x_\ell^* \\
\& \ \frac{\partial v(p,w)}{\partial w} &= \frac{\partial}{\partial w}\mathcal{L}(x^*,\mu^*,\lambda^*;p,w) \\
&= \frac{\partial}{\partial w}\left[u(x^*) + \sum_{\ell=1}^{L}\mu_\ell^* x_\ell^* - \lambda^*\left(\sum_{\ell=1}^{L} p_\ell x_\ell^* - w\right)\right] = \lambda^*
\end{aligned}$$

(b) Shephard's Lemma

$$h_\ell(p,u) = \frac{\partial e(p,u)}{\partial p_\ell}$$

Proof: By the envelope theorem

$$\begin{aligned}
\frac{\partial e(p,u)}{\partial p_\ell} &= -\frac{\partial}{\partial p_\ell}\mathcal{Z}(x^*,\mu^*,\gamma^*;p,u) \\
&= \frac{\partial}{\partial p_\ell}\left[\sum_{\ell=1}^{L} p_\ell x_\ell^* - \sum_{\ell=1}^{L}\mu_\ell^* x_\ell^* + \gamma^*(u - u(x^*))\right] \\
&= x_\ell^*
\end{aligned}$$

(c) Slutsky Equation: Obtained by differentiating w.r.t p_k the identity

$$h_\ell(p,u) \equiv x_\ell(p, e(p,u))$$

$$\begin{aligned}
\frac{\partial h_\ell(p,u)}{\partial p_k} &= \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} \times \frac{\partial e(p,u)}{\partial p_k} \\
&= \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w)\text{, where } w = e(p,u).
\end{aligned}$$

Or in Matrix notation

$$\underbrace{D_p h(p,u)}_{L\times L} = \underbrace{D_p x(p,w)}_{L\times L} + \underbrace{[D_w x(p,w)]^{\mathrm{T}}}_{L\times 1}\underbrace{x(p,w)^{\mathrm{T}}}_{1\times L}$$