3. PRODUCTION THEORY

Ref: MWG Chapter 5

Productive units – "firms"

- corporations, other legally recognized businesses
- productive possibilities of individuals or households
- potential productive units that are never actually organized.

"Black box" – able to transform inputs into outputs.

3.1 Production Sets

production vector $y \in \mathbb{R}^L$

$$\text{e.g. } y = \begin{pmatrix} -5 \\ 2 \\ -6 \\ 3 \\ 0 \end{pmatrix} \in \mathbb{R}^5$$

Menu of all possible production vectors constitutes Y, *the production set.*

a) *Transformation frontier* $Y = \left\{ y \in \mathbb{R}^L \mid F(y) \leq 0 \right\}$

$F(y) = 0$ means y element of *boundary* of Y.

$$\text{MRT}_{\ell k}(\bar{y}) = \frac{\partial F(\bar{y}) / \partial y_\ell}{\partial F(\bar{y}) / \partial y_k}$$

Notice

$$\frac{\partial F(\bar{y})}{\partial y_k} \frac{dy_k}{dy_\ell} + \frac{\partial F(\bar{y})}{\partial y_\ell} = 0 \text{ so, } \frac{dy_k}{dy_\ell} = -\text{MRT}_{\ell k}(\bar{y})$$

b) *Production function.* $q = f(z)$

$$Y = \left\{ (-z_1, \ldots - z_{L-1}, q) \mid \begin{array}{c} q - f(z_1, \ldots, z_{L-1}) \leq 0, \\ z_\ell \geq 0,\ \ell = 1, \ldots L-1 \end{array} \right\}$$

Holding level of output fixed:

$$\text{MRTS} = \frac{\partial f(\bar{z}) / \partial z_\ell}{\partial f(\bar{z}) / \partial z_k}$$

additional amount of input k that must be used to keep output fixed at $\bar{q} = f(\bar{z})$, when amount of input ℓ decreased marginally.

3.2 Properties of Production Sets (see pp 130-155)

1. free disposal
2. non-increasing returns to scale
3. non-decreasing RTS
4. constant RTS

3.3 Profit Maximization

Profit Max. Pblm (PMP) $\max\limits_{y \in Y}\ p.y$ or $\max\limits_{y}\ p.y$ s.t. $F(y) \leq 0$

Profit function $\pi(p) = \max_{y \in Y}\ p.y$

Supply correspondence $y(p) = \{y \in Y \mid p.y = \pi(p)\}$

Ex. 3.1

$$\begin{aligned} Y &= \left\{ y \in \mathbb{R}^2 \mid y_1 + y_2 \leq 0,\ y_1 \leq 0 \right\} \\ \pi(p) &= \begin{cases} 0 & \text{if } p_2 \leq p_1 \\ \infty & \text{if } p_2 > p_1 \end{cases} \\ y(p) &= \begin{cases} 0 & \text{if } p_2 \leq p_1 \\ \left\{ y \in \mathbb{R}^2 \mid y_2 = -\ y_1 \geq 0 \right\} & \text{if } p_2 = p_1 \\ \text{undefined} & \text{if } p_2 > p_1 \end{cases} \end{aligned}$$

First order approach

(i) Transformation frontier

$$\max_{y}\ p.y \text{ s.t. } F(y) \leq 0 \Rightarrow \mathcal{L} = p.y - \lambda F(y)$$

$$\text{FONC } y_\ell : p_\ell = \lambda \frac{\partial F(y^*)}{\partial y_\ell}$$

or in matrix notation

$$p = \lambda \nabla F(y^*)$$

(ii) Production function

$$\max_{z \geq 0}\ pf(z) - w.z$$

$$\text{FONC } z_\ell : p \frac{\partial f(z^*)}{\partial z_\ell} \leq w_\ell \ (= w_\ell, \text{ if } z_\ell^* > 0)$$

or in matrix notation

$$p\nabla f(z^*) \leq w \text{ and } (p\nabla f(z^*) - w).z^* = 0$$

Properties of the Profit Function

Given Y is closed and satisfies free disposal.

1. $\pi(p)$ is homogeneous of degree one in p, i.e. $\pi(\alpha p) \equiv \alpha\pi(p)\ \forall\ \alpha > 0$
2. $\pi(p)$ is convex in p
3. $y(p)$ is homogenous of degree zero.
4. If Y is convex, then
   (a) $y(p)$ is a convex set for all p
   (b) $Y = \left\{y \in \mathbb{R}^L : p.y \leq \pi(p) \text{ for all } p \gg 0\right\}$
5. If $Y = \left\{y \in \mathbb{R}^L : F(y) \leq 0\right\}$ and F is strictly convex, then $y(p)$ is single-valued for all $p \gg 0$.
6. (Hotelling's lemma) If $y(p)$ is single-valued at $p = \bar{p}$, then $\pi(\cdot)$ is differentiable at $\bar{p}$ and $\nabla\pi(\bar{p}) = y(\bar{p})$
7. If $y(\cdot)$ is a fn differentiable at $\bar{p}$, then $\mathrm{D}_p\, y(\bar{p}) = \pi_{pp}(\bar{p})$ is a symmetric and positive semidefinite matrix with

$$\mathrm{D}_p\, y(\bar{p})\, \bar{p} = 0$$

3.4 Cost Minimization

Implication of π-max: no way to produce same amount of outputs at lower total input cost. I.e. *cost minimization* is *necessary* condition for π-max.

1. Leads to no. of results and construction that are technically useful.
2. When firm is *not* a price-taker in output market, no longer use profit fn for analysis. But if price-taker in input markets, results flowing from cost minimization problem (CMP) still valid.

   Single output case:- CMP and cost function

$$c(w, q) = \min_{z \geq 0} \; w.z \text{ s.t. } f(z) \geq q$$

Solution

$$z(w, q) \text{ – } \textit{conditional} \text{ factor demand correspondence}$$

First order approach

$$\max_{z \geq 0} -w.z \text{ s.t. } q - f(z) \leq 0 \Rightarrow \mathcal{Z} = -w.z - \gamma(q - f(z))$$

$$\text{FONC } z_\ell : w_\ell \geq \gamma \frac{\partial f(z^*)}{\partial z_\ell} \;\left(= \gamma \frac{\partial f(z^*)}{\partial z_\ell} \text{ if } z_\ell^* > 0 \right)$$

Or in matrix notation

$$w \geq \gamma \nabla f(z^*) \text{ and } (w - \gamma \nabla f(z^*)).z^* = 0$$

Notice, from FONC

$$\frac{\partial f(z^*)/\partial z_\ell}{\partial f(z^*)/\partial z_k} \equiv \text{MRTS}_{\ell k} = \frac{w_\ell}{w_k}$$

As usual, Lagrange multiplier γ may be interpreted as the marginal value of "tightening" the constraint $f(z^*) \geq q$. Hence

$$\gamma = \frac{\partial}{\partial q} c(w, q)$$

is the *marginal cost of production*.

Properties of Cost Function

1. $c(w,q)$ is homogeneous of degree one in w and non-decreasing in q.
2. $c(w,q)$ is a concave function of w.
3. If the sets
$$\{z \geq 0 : f(z) \geq q\}$$
are convex for every $q \geq 0$, then
$$Y = \{(-z,q) : w \cdot z \geq c(w,q) \text{ for all } w \gg 0\}$$
4. If $\{z \geq 0 : f(z) \geq q\}$ is convex (respectively, strictly convex), then $z(w,q)$ is a convex set (respectively, single-valued).

5. (Shephard's lemma) If $z(\bar{w},\bar{q})$ is single-valued, then $c(\bar{w},\bar{q})$ is differentiable wrt w at $(\bar{w},\bar{q})$ and
$$\nabla_w c(\bar{w},\bar{q}) = z(\bar{w},\bar{q})$$
6. If $z(w,q)$ is differentiable wrt w at $(\bar{w},\bar{q})$, then
$$\mathrm{D}_w\, z(\bar{w},\bar{q}) = \mathrm{D}_{ww}\, c(\bar{w},\bar{q})$$
is a symmetric and negative semi-definite matrix with
$$\mathrm{D}_w\, z(\bar{w},\bar{q})\, \bar{w} = 0$$
7. If $f(z)$ is homogeneous of degree 1 (i.e. exhibits CRS) then $c(w,q)$ and $z(w,q)$ are homogeneous of degree 1 in q.
8. If $f(z)$ is concave, then $c(w,q)$ is a convex function of q (in particular, marginal cost is non-decreasing in q).

3.5 Geometry of Cost & Supply in Single-Output Case

(MWG pp143-147).

3.6 Aggregation.

J production units $Y^1, \ldots, Y^J$
For each Y^j, let $\pi^j(p)$ and $y^j(p)$ be the assoc. profit function and supply correspondence.

Aggregate supply correspondence

$$\begin{aligned} y(p) &= \sum_{j=1}^{J} y^j(p) \\ &= \left\{ y \in \mathbb{R}^L \mid y = \sum_{j=1}^{J} y^j \text{ for some } y^j \in y^j(p) \right\} \end{aligned}$$

If each $y^j(.)$ is a differentiable function,

then $D_p y^j(p)$ is a *symmetric PSD* matrix, hence

$$D_p y(p) = \sum_{j=1}^{J} D_p y^j(p) \text{ is a symmetric PSD matrix.}$$

and we have an *aggregate law of supply*

$$(p - \widehat{p}) . (y(p) - y(\widehat{p})) \geq 0$$

3.6.1 Representative Producer.

Given J production units $Y^1, \ldots, Y^J$, define *aggregate production set*

$$\begin{aligned} Y &= Y^1 + \ldots + Y^J \\ &= \left\{ y \in \mathbb{R}^L \mid y = \sum_{j=1}^{J} y^j \text{ for some } y^j \in Y^J \right\} \end{aligned}$$

Let $\pi^*(p)$ and $y^*(p)$ be the profit fn and supply correspondence associated with the aggregate production set.

Proposition 3.6.1 For all $p \gg 0$ we have:

1. $\pi^*(p) = \sum_{j=1}^{J} \pi^j(p)$
2. $y^*(p) = \sum_{j=1}^{J} y^j(p)$.

Decentralization Result: to find solution of aggregate profit max. problem for given prices p, it is enough to add solutions of corresponding individual problems.

Implication in single-output case:
If firms are max profits facing output price p and factor prices w, then their supply behavior maximizes aggregate profits. Hence if

$$q = \sum_{j=1}^{J} q^j$$

is aggregate output produced by firms, then total cost of production equals $c(w, q)$, the *value* of the aggregate cost function. Thus, allocation of production of output level q among the firms is cost minimizing.

3.6.2 Efficiency

A netput vector $y \in Y$ is *efficient* if there is no $\widehat{y} \in Y$ s.t. $\widehat{y} \geq y$ & $\widehat{y} \neq y$.

Proposition (1^{st} Fundamental Welfare Theorem - production side)
If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient.

Proof. Suppose the contrary. That is, suppose there is $\widehat{y} \in Y$ s.t. $\widehat{y} \geq y$ and $\widehat{y} \neq y$. Because $p \gg 0$, it follows that $p.\widehat{y} > p.y$, contradicting assumption that y is profit maximizing.

Remark: FFWT is valid even if production set is non-convex.

Proposition (2^{nd} Fundamental Welfare Theorem - production side)
Suppose Y is convex, closed and satisfies free-disposal property. Then every efficient production $y \in Y$ is a profit-maximizing production for some non-zero price vector $p \geq 0$.

Proof. Application of separating hyperplane theorem for convex sets.

3.7 Price-taking and Profit Maximizing.

- Assumption of preference maximization is *natural* objective for theory of the consumer.

- Profit maximization not so self-evident.
  - What about sale revenue? market share?
  - size of firm? size of workforce?

- Ideally objective of the firm should emerge from objectives of individuals who control it.
  - firm with single owner has well-defined objective.
    - only issue: whether this objective coincides with profit max.
  - multiple owners – potential for conflicting objectives.

Q. When is profit-maximization *unanimously agreed upon* objective?

A. Suppose firm described by production set Y and owned by consumers. Let θ^i be share of firm owned by consumer i, where $\sum_i \theta^i = 1$. If production decision is $y \in Y$, then i with utility fn u^i achieves utility level

$$\max_{x^i \in X} u^i\left(x^i\right) \text{ s.t. } p.x^i \leq w^i + \theta^i p.y^i$$

Follows at any given price vector p, consumer-owners *unanimously* prefer firm to implement production plan $\widehat{y} \in Y$ instead of $y \in Y$ whenever

$$p.\widehat{y} > p.y$$

Notice we are assuming:

1. prices fixed and do not depend on actions of the firm.
2. profits are not uncertain.
3. managers can be controlled by owners.

1. If prices depend upon production of firm, objective of owners may depend on their tastes as consumers.

   e.g. Firm produces good 2 using good 1 as input according to production function $f(.)$. Normalize $p_1 = 1$ suppose $p_2 = p(q)$.
   Suppose further that owner's only wealth is from profits of firm.

   (a) if care only about consumption of good 1

   $$\max_{z \geq 0} \; p(f(z)) f(z) - z$$

   (b) if care only about consumption of good 2

   $$\max_{z \geq 0} \frac{p(f(z)) f(z) - z}{p(f(z))}$$

   2 problems in general have different solutions.

2. If output of firms is random, crucial to distinguish between whether output is sold *before* or *after* uncertainty is resolved.

   (a) if after, then π uncertain at time of production decision, so risk preferences relevant.
   (b) if before (e.g. futures market for agricultural products), then risk borne by buyer,
   so unanimity of profit maximization goes through.