2. CHOICE UNDER UNCERTAINTY

Ref: MWG Chapter 6

Subjective Expected Utility Theory

Elements of decision under uncertainty

Under uncertainty, the DM is forced, in effect, to gamble.

A right decision consists in the choice of the best possible bet,
not simply in whether it is won or lost after the fact.

Two essential characteristics:

1. A *choice* must be made among various possible courses of actions.
2. This choice or sequence of choices will ultimately lead to some *consequence*, but DM cannot be sure in advance what this consequence will be, because it depends not only on his or her choice or choices but on an unpredictable *event*.

Simple and Compound Lotteries

- $X =$ (finite) set of outcomes (what DM cares about).
- $\mathcal{L}$ set of simple lotteries (prob. distributions on X with finite support). A lottery L in $\mathcal{L}$ is a *fn* $L : X \to \mathbb{R}$, that satisfies following 2 properties:

  1. $L(x) \geq 0$ for every $x \in X$.
  2. $\sum_{x \in X} L(x) = 1$.

Examples: Take $X = \{-1000, -900, \ldots, -100, 0, 100, 200, \ldots, 900, 1000\}$

1. A 'fair' coin is flipped and the individual wins \$100 if heads, wins nothing if tails

$$L_1(x) = \begin{cases} 1/2 & \text{if } x \in \{0, 100\} \\ 0 & \text{if } x \notin \{0, 100\} \end{cases}$$

2. Placing a bet of \$100 on black on a (European) roulette wheel

$$L_2(x) = \begin{cases} 18/37 & \text{if } x = 100 \\ 19/37 & \text{if } x = -100 \\ 0 & \text{if } x \notin \{-100, 100\} \end{cases}$$

3. A pack of 52 playing cards is shuffled. Win \$200 if the top card is an Ace, lose \$500 if the top card is the Queen of Spades otherwise no change in wealth.

$$L_3 = \begin{cases} 1/13 & \text{if } x = 200 \\ 47/52 & \text{if } x = 0 \\ 1/52 & \text{if } x = -500 \\ 0 & \text{if } x \notin \{-500, 0, 200\} \end{cases}$$

4. A 'balanced' die is rolled. Win \$100 if number on top is even & win nothing otherwise.

$$L_4(x) \equiv L_1(x) = \begin{cases} 1/2 & \text{if } x \in \{0, 100\} \\ 0 & \text{if } x \notin \{0, 100\} \end{cases}$$

A *compound lottery* is a two-stage lottery in which the outcomes from the first-stage randomization are themselves lotteries.

Formally, a compound lottery is a fn $\mathbf{C} : \mathcal{L} \to \mathbb{R}$, that satisfies the following 2 properties:

1. $\mathbf{C}(L) \geq 0$ for every $L \in \mathcal{L}$, with strict inequality for only finitely many lotteries L.

2. $\sum_{L \in \mathcal{L}} \mathbf{C}(L) = 1$.

Example: A 'fair' coin is flipped and the individual then plays out L_2 if heads and L_3 if tails.

$$\mathbf{C}_1(L) = \begin{cases} 1/2 & \text{if } L \in \{L_2, L_3\} \\ 0 & \text{if } L \notin \{L_2, L_3\} \end{cases}$$

REDUCTION: 'Multiply through' 1^{st}-stage prob. *to reduce* a compound lottery to a one-stage lottery. I.e., if $\alpha_1, \ldots, \alpha_n$ are the prob. of the possible 2^{nd}-stage lotteries $L_1, \ldots, L_n$ then the *reduction* is the lottery

$$\alpha_1 L_1 + \alpha_2 L_2 + \ldots + \ \alpha_n L_n$$

Example cont.: Reduction of $\mathbf{C}_1(L)$ is lottery $R_1 = (1/2)\,L_2 + (1/2)\,L_3$,

$$\text{i.e., } R_1(x) = \begin{cases} 1/26 & \text{if } x = 200 \\ 9/37 & \text{if } x = 100 \\ 47/104 & \text{if } x = 0 \\ 19/74 & \text{if } x = -100 \\ 1/104 & \text{if } x = -500 \\ 0 & \text{otherwise} \end{cases}$$

Consequentialism: assume individual indifferent between any compound lottery and the associated reduced lottery.
We can also see that the set of lotteries is a 'mixture space'.
If L & L' are lotteries in $\mathcal{L}$ then for any α in $[0,1]$, $\alpha L + (1-\alpha)\,L'$

is the lottery $L''(x) = \alpha L(x) + (1-\alpha)\,L'(x)$

To see that L'' is indeed a lottery, notice that:

1. $L''(x) = \alpha L(x) + (1-\alpha)\,L'(x) \geq 0$, for every $x \in X$.
2. $\sum_{x\in X} L''(x) = \sum_{x\in X} [\alpha L(x) + (1-\alpha)\,L'(x)]$
   $= \alpha \sum_{x\in X} L(x) + (1-\alpha)\sum_{x\in X} L'(x) = \alpha + (1-\alpha) = 1.$

Further notation: for any $x \in X$,
let δ_x denote the *degenerate* lottery $(x,1) \in \mathcal{L}$, i.e.

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

Hence for any lottery L in $\mathcal{L}$ we have $L = \sum_{x\in X} L(x)\,\delta_x$.

"States-of-nature" model

- $X =$ set of outcomes (what DM cares about)
- $\mathcal{L}$ set of simple lotteries
  (probability distributions on X with finite support).
- $S =$ set of states
  (uncertain factors beyond the control of the DM)
- $A =$ set of acts (what the DM controls or chooses)
- $\succsim$ defined over acts

Formally, we will take A to be the set of functions

$$a : S \to \mathcal{L}$$

with *finite range*. That is, any act a may be expressed in a form

$$[L_1, E_1; \ldots ; L_n, E_n]$$

where $\{E_1, \ldots , E_n\}$ forms a *finite* partition of the state space.

An act that maps each state to a degenerate lottery may be viewed as a purely subjectively uncertain act.

$$[\delta_{x_1}, E_1; \ldots ; \delta_{x_n}, E_n]$$

Set of acts is also a 'mixture space'.

For any pair of acts a and a', $\alpha a + (1-\alpha) a'$ is the act $a'' : S \to \mathcal{L}$, for which

$$a''(s) = \alpha a(s) + (1-\alpha) a'(s)$$

State: complete specification of the past, present and future configuration of the world, except for those details that are part of the DM's actions.

Often can analyze situation in terms of a finite partition of the state space

$$\{E_1, \ldots, E_n\}$$

Set of mutually exclusive and exhaustive events.

Example: Jones faces choice between current employment or doing MBA.

- Jones has the choice between two possible 'acts': $leave$ and $stay$.
- Three outcomes
  1. $x =$ stay in current employment
  2. $0 =$ incur costs of undertaking MBA, but after graduating only get job similar to the one he had before.
  3. $M =$ incur costs of undertaking MBA and after graduating land extremely well-paying and exciting job.
- The event E in which Jones obtains the high-paying job if he has chosen to $leave$.

$$leave(s) = \begin{cases} \delta_M & \text{if } s \in E \\ \delta_m & \text{if } s \notin E \end{cases} \quad stay(s) = \begin{cases} \delta_x & \text{if } s \in E \\ \delta_x & \text{if } s \notin E \end{cases}$$

Whether we have $leave \succsim stay$ or $stay \succsim leave$ would seem to depend on two separate considerations:

1. How good *Jones feels* the chances of obtaining the high-paying job would have to be to make it worth his while to leave his current employment;
2. How good *in his opinion* the chances of obtaining the high-paying job actually are.

Jones's answers to questions of type 1 *quantify* his *personal preference* for x relative to 0 and M.

Jones's answers to questions of type 2 *quantify* his *personal judgement* concerning the relative strengths of the factors that favor and oppose certain events.

If he behaves *reasonably* then he should choose the solution of the problem which is *consistent* with his personal preference and his personal judgement.

1. How might Jones quantify his preference for x relative to 0 and M?
2. How might Jones quantify his judgement for likelihood of event E?

We know $\delta_M \succ \delta_x \succ \delta_m$, so set $U(\delta_M) := 1$ and $U(\delta_m) := 0$.

Let u_x be the unique probability for which

$$u_x \delta_M + (1 - u_x)\,\delta_m \sim \delta_x.$$

Let π_E be the unique probability for which

$$\pi_E \delta_M + (1 - \pi_E)\,\delta_m \sim \left[\begin{array}{ll} \delta_M & \text{on } E \\ \delta_m & \text{on } S - E \end{array}\right]$$

and set

$$\begin{aligned} U(stay) &:= u_x U(\delta_M) + (1 - u_x)\,U(\delta_m) \\ U(leave) &:= \pi_E U(\delta_M) + (1 - \pi_E)\,U(\delta_m) \end{aligned}$$

Hence we have

$$leave \succsim stay \Leftrightarrow \pi_E \geq u_x$$

Principles of Choice Behavior.

Axioms for $\succsim$.

For ease of exposition, suppose there exists two outcomes M and m, such that $\delta_M \succ \delta_m$ and for all $x \in X$, $\delta_M \succsim \delta_x \succsim \delta_m$.

Ordering Axiom $\succsim$ is complete and transitive.

Archimedean Axiom For any three acts, a, a' and a'', for which $a' \succ a \succ a''$, there exists numbers α and β, both in $(0,1)$ such that

$$\alpha a' + (1-\alpha)\,a'' \succ a \succ \beta a' + (1-\beta)\,a''.$$

The Archimedean axiom rules out a *lexicographic* preference for certainty. Thus it plays a similar role to that played by the continuity axiom in decision making under certainty: ruling out discontinuous 'jumps' in the preference relation.

Example (MWG p171) Suppose

$$M = \text{'beautiful \& uneventful trip by car'}$$
$$x = \text{'staying at home'}$$
$$m = \text{'death by car crash'}.$$

Set $a' := \delta_M$, $a := \delta_x$ and $a'' := \delta_m$. If

$$a' \succ a \succ a''$$

then there exists sufficiently large $\alpha < 1$, such that

$$\begin{aligned} \alpha a' + (1-\alpha)\,a'' &\succ a \\ \text{i.e. } [M,\alpha; m, 1-\alpha] &\succ [x,1] \end{aligned}$$

REMARK From people's *revealed behavior*, axiom is quite sound empirically.

Independence Axiom: For all $a, a', a'' \in A$ and $\lambda \in (0,1)$ we have

$$a \succsim a' \Leftrightarrow \lambda a + (1-\lambda)a'' \succsim \lambda a' + (1-\lambda)a''$$

Embodies a 'substitution' principle and a reduction of compound lotteries principle.

E.g. Akbar has free international round-trip ticket and is planning to use it for his winter vacation. His preferred destinations, Hawaii and Madrid, are sold out. So he makes a reservation for Cancun. He can also choose to be wait-listed for Hawaii or Madrid, but not both. If he decides to get on the waiting list for Hawaii, then he has a fifty percent chance of ultimately getting a reservation otherwise he will go to Cancun. If he decides to get on the waiting list for Madrid, however, the situation is completely different. First, the probability of getting a reservation for Madrid is only 1/4 rather than 1/2 and secondly, to get on this waiting list, he has to *drop* his reservation for Cancun. If he doesn't get a reservation for Madrid, there is a 2/3 chance he can get back his reservation for Cancun, but there is a 1/3 chance he will only be able to get a reservation for Toronto.

$X = \{$C[ancun], H[awii], M[adrid], T[oronto]$\}$. Suppose for Akbar,

$$\delta_M \succ \delta_H \succ \delta_C \succ \delta_T$$

Hence by independence

$$\frac{1}{2}\delta_M + \frac{1}{2}\delta_C \succ \frac{1}{2}\delta_H + \frac{1}{2}\delta_C$$

Furthermore preference between

$$\frac{1}{2}\delta_H + \frac{1}{2}\delta_C \text{ and } \frac{1}{4}\delta_M + \frac{3}{4}\left(\frac{2}{3}\delta_C + \frac{1}{3}\delta_T\right)$$

is determined by preference between

$$\delta_H \text{ and } \frac{1}{2}\delta_M + \frac{1}{2}\delta_T.$$

Since

$$\begin{aligned}\frac{1}{4}\delta_M + \frac{3}{4}\left(\frac{2}{3}\delta_C + \frac{1}{3}\delta_T\right) &= \frac{1}{4}\delta_M + \frac{1}{4}\delta_T + \frac{1}{2}\delta_C \\ &= \frac{1}{2}\left(\frac{1}{2}\delta_M + \frac{1}{2}\delta_T\right) + \frac{1}{2}\delta_C\end{aligned}$$

Further notation: for any act a, any lottery L and any event E, let $L_E a$ denote the act a' where

$$a'(s) = \begin{cases} L & \text{if } s \in E \\ a(s) & \text{if } s \notin E \end{cases}$$

***State-Independence Axiom*:** For any pair of lotteries L and L', and any event E, such that $(\delta_M)_E(\delta_m) \succ \delta_m$,

$$L \succsim L' \Leftrightarrow L_E a \succsim L'_E a$$

The unconditional preference between any pair of lotteries is the same as the preference between those lotteries conditional on any non-null event having obtained.

A function $U : \mathcal{L} \to \mathbb{R}$ is affine, if for all $L, L' \in \mathcal{L}$ and $\alpha \in [0, 1]$,

$$U(\alpha L + (1 - \alpha) L') = \alpha U(L) + (1 - \alpha) U(L').$$

Fact: If $U : \mathcal{L} \to \mathbb{R}$ is affine, then there exists a function $u : X \to \mathbb{R}$ such that for all $L = [x_1, p_1; \ldots; x_m, p_m]$, $U(L) = \sum_{i=1}^{m} p_i u(x_i)$.

The big result:

THEOREM: Suppose there exists two outcomes M and m, such that $\delta_M \succ \delta_m$ and for all $x \in X$, $\delta_M \succsim \delta_x \succsim \delta_m$. Then the following are equivalent:

1. The preference relation $\succsim$ satisfies the *Ordering*, *Archimedean*, *Independence* and *State-Independence Axioms*.
2. The preference relation $\succsim$ admits a *subjective expected utility representation*. That is, there exists a unique probability measure π and a unique affine function $U : \mathcal{L} \to [0,1]$, with $U(\delta_m) = 0$ and $U(\delta_M) = 1$, such that for any pair of acts

$$a = [L_1, E_1; \ldots; L_n, E_n] \;\&\; a' = [L_1', E_1'; \ldots; L_{n'}, E_{n'}]$$

$$a \succsim a' \Leftrightarrow \sum_{i=1}^{n} \pi(E_i)\, U(L_i) \geq \sum_{j=1}^{n'} \pi\left(E_j'\right) U\left(L_j'\right)$$

Proof of Theorem:

(2) implies (1) (Exercise.)

(1) implies (2)

Preliminary Results The axioms imply that $\succcurlyeq$ exhibits the following properties.

Mixture Monotonicity For any $a, a' \in A$, such that $a \succ a'$, and any $\alpha \in (0,1)$,

$$a \succ \alpha a + (1-\alpha)\, a' \succ a'$$

Proof of Mixture Monotonicity: By independence
$a = \alpha a + (1-\alpha)\, a \succ \alpha a + (1-\alpha)\, a'$
and $\alpha a + (1-\alpha)\, a' \succ \alpha a' + (1-\alpha)\, a' = a$. □

Mixture Solvability For any $a, a', a'' \in A$, for which $a' \succ a \succ a''$, there exists a unique $\alpha \in (0,1)$ such that

$$\alpha a' + (1-\alpha)\, a'' \sim a$$

Proof of Mixture Solvability: Consider the sets

$$\alpha^{+} = \{\alpha \in [0,1] : \alpha a' + (1-\alpha)\, a'' \succ a\}\text{, and}$$
$$\alpha^{-} = \{\alpha \in [0,1] : a \succ \alpha a' + (1-\alpha)\, a''\}.$$

From *Mixture Monotonicity* it follows that both α^{+} and α^{-} are non-empty, non-intersecting and connected subsets of $[0,1]$. Moreover, the greatest lower bound for α^{+} equals the least upper bound for α^{-}. Denote this number by $\bar{\alpha}$.

Thus it must be the case that one of the following hold: (i) $\bar{\alpha} \in \alpha^{+}$ and $\bar{\alpha} \notin \alpha^{-}$, or (ii) $\bar{\alpha} \notin \alpha^{+}$ and $\bar{\alpha} \in \alpha^{-}$, or (iii) $\bar{\alpha} \notin \alpha^{+}$ and $\bar{\alpha} \notin \alpha^{-}$. So first suppose $\bar{\alpha} \in \alpha^{+}$ and $\bar{\alpha} \notin \alpha^{-}$, that is,

$$\bar{\alpha} a' + (1-\bar{\alpha})\, a'' \succ a \succ a''.$$

But then it follows that for any β in $(0,1)$, we have

$$\begin{aligned} a &\succ \beta\left(\bar{\alpha} a' + (1-\bar{\alpha})\, a''\right) + (1-\beta)\, a'' \\ &= \beta\bar{\alpha} a' + (1-\beta\bar{\alpha})\, a'' \text{ (since } \beta\bar{\alpha} \in \alpha^{-}\text{)} \end{aligned}$$

a violation of the Archimedean axiom. By similar reasoning we also get a violation of the Archimedean axiom if we assume $\bar{\alpha} \notin \alpha^{+}$ and $\bar{\alpha} \in \alpha^{-}$. Hence we must have $\bar{\alpha} \notin \alpha^{+}$ and $\bar{\alpha} \notin \alpha^{-}$, and hence by completeness we have $\bar{\alpha} a' + (1-\bar{\alpha})\, a'' \sim a$, as required. □

We are now in a position to show **(1)** implies **(2)**, by explicitly constructing the SEU-representation for $\succsim$. We proceed by first deriving an Expected Utility representation for the preference relation restricted to the set of constant acts. That is, we construct the affine real-valued function U defined on $\mathcal{L}$. In the second step, we use this U to calibrate the decision weights on events to construct the probability measure π defined on S, that enables us to extend the representation to the entire set of acts.

Step 1. Constructing the EU-Representation on $\succsim$ restricted to $\mathcal{L}$ (the set of *constant acts*).

Set $U(\delta_M) := 1$ and $U(\delta_m) := 0$. For any $x \in X$ set $U(\delta_x) := \beta$, where, by *Mixture Solvability*, β is the unique solution to $\beta\delta_M + (1-\beta)\delta_m \sim \delta_x$. For any $L = \sum_{i=1}^{m} \alpha_i \delta_{x_i} \in \mathcal{L}$ we can apply *Independence* and transitivity of indifference (*Ordering*) m times to obtain

$$\begin{aligned} L &\sim \alpha_1 \left(U\left(\delta_{x_1}\right)\delta_M + \left(1 - U\left(\delta_{x_1}\right)\right)\delta_m\right) + \sum_{i=2}^{m} \alpha_i \delta_{x_i} \\ &\sim \cdots \sim \sum_{i=1}^{m} \alpha_i \left(U\left(\delta_{x_i}\right)\delta_M + \left(1 - U\left(\delta_{x_i}\right)\right)\delta_m\right) \\ &= \left(\sum_{i=1}^{m} \alpha_i U\left(\delta_{x_i}\right)\right)\delta_M + \left(1 - \left(\sum_{i=1}^{m} \alpha_i U\left(\delta_{x_i}\right)\right)\right)\delta_m \end{aligned}$$

Hence for any pair of constant acts $L = \sum_{i=1}^{m} \alpha_i \delta_{x_i}$ and $L' = \sum_{j=1}^{m'} \beta_j \delta_{x_j}$, transitivity of preference (*Ordering*) implies $L \succsim L'$ iff

$$\begin{aligned} & \left(\sum_{i=i}^{m} \alpha_i U\left(\delta_{x_i}\right) \right) \delta_M + \left(1 - \left(\sum_{i=i}^{m} \alpha_i U\left(\delta_{x_i}\right) \right) \right) \delta_m \\ \succsim \quad & \left(\sum_{j=i}^{m'} \beta_j U\left(\delta_{x_j}\right) \right) \delta_M + \left(1 - \left(\sum_{j=1}^{m'} \beta_j U\left(\delta_{x_j}\right) \right) \right) \delta_m . \end{aligned}$$

But by *Mixture Monotonicity* this holds if and only if

$$\left(\sum_{i=i}^{m} \alpha_i U\left(\delta_{x_i}\right) \right) \geq \left(\sum_{j=i}^{m'} \beta_j U\left(\delta_{x_j}\right) \right) .$$

Hence the affine function

$$U\left(\sum_{i=1}^{m} \alpha_i \delta_{x_i} \right) = \sum_{i=1}^{m} \alpha_i U\left(\delta_{x_i}\right)$$

represents $\succsim$ restricted to the set of constant acts.

Step 2. Constructing the SEU-Representation for $\succsim$.

Fix any a in A and express it in a form $[L_1,\ E_1; \ldots;\ L_n,\ E_n]$, where $L_i \succsim L_{i+1}$, for all $i = 1, \ldots, i-1$. For each $i = 1, \ldots, n$, it follows from Step 1 that there is a unique number $U(L_i) \in [0,1]$, for which

$$L_i \sim U(L_i)\,\delta_M + (1 - U(L_i))\,\delta_m.$$

For each $i = 1, \ldots, n-1$, it follows from *Mixture Solvability* that there exists a unique π_i satisfying

$$[\delta_M \text{ on } E_1 \cup \ldots \cup E_i;\ \delta_m \text{ on } E_{i+1} \cup \ldots \cup E_n] \sim \pi_i \delta_M + (1 - \pi_i)\,\delta_m.$$

From *Mixture Monotonicity* it follows $1 \geq U(L_1) \geq \ldots \geq U(L_n) \geq 0$.

From *State-independence* it follows $0 \leq \pi_1 \leq \ldots \leq \pi_{n-1} \leq 1$.

By applying *State-independence* n times we obtain

$$a = \begin{bmatrix} L_1 & \text{on } E_1 \\ \vdots & \vdots \\ L_n & \text{on } E_n \end{bmatrix} \sim \begin{bmatrix} U(L_1)\,\delta_M + (1 - U(L_1))\,\delta_m & \text{on } E_1 \\ \vdots & \vdots \\ U(L_n)\,\delta_M + (1 - U(L_n))\,\delta_m & \text{on } E_n \end{bmatrix}$$

$$= (1 - U(L_1)) \begin{bmatrix} \delta_m & \text{on } E_1 \\ \delta_m & \text{on } E_2 \\ \vdots & \vdots \\ \delta_m & \text{on } E_{n-1} \\ \delta_m & \text{on } E_n \end{bmatrix} + (U(L_1) - U(L_2)) \begin{bmatrix} \delta_M & \text{on } E_1 \\ \delta_m & \text{on } E_2 \\ \vdots & \vdots \\ \delta_m & \text{on } E_{n-1} \\ \delta_m & \text{on } E_n \end{bmatrix} + \cdots$$

$$+ (U(L_{n-1}) - U(L_n)) \begin{bmatrix} \delta_M & \text{on } E_1 \\ \delta_M & \text{on } E_2 \\ \vdots & \vdots \\ \delta_M & \text{on } E_{n-1} \\ \delta_m & \text{on } E_n \end{bmatrix} + U(L_n) \begin{bmatrix} \delta_M & \text{on } E_1 \\ \delta_M & \text{on } E_2 \\ \vdots & \vdots \\ \delta_M & \text{on } E_{n-1} \\ \delta_M & \text{on } E_n \end{bmatrix}$$

By applying *Independence* $n-1$ times we have a is indifferent to:

$$(1-U(L_1))\,\delta_m + (U(L_1)-U(L_2))\left[\pi_1\delta_M + (1-\pi_1)\,\delta_m\right]$$

$$+\,(U(L_2)-U(L_3))\left[\pi_2\delta_M + (1-\pi_2)\,\delta_m\right]$$

$$+\ldots+(U(L_{n-1})-U(L_n))\left[\pi_{n-1}\delta_M + (1-\pi_{n-1})\,\delta_m\right] + U(L_n)\,\delta_M$$

$$= \left[\sum_{i=1}^{n-1}(U(L_i)-U(L_{i+1}))\,\pi_i + U(L_n)\right]\delta_M + \left[1-U(L_n)-\sum_{i=1}^{n-1}(U(L_i)-U(L_{i+1}))\,\pi_i\right]\delta_m.$$

Hence if we take any given pair of acts

$$a = \begin{bmatrix} L_1 & \text{on } E_1 \\ \vdots & \vdots \\ L_n & \text{on } E_n \end{bmatrix} \text{ and } a' = \begin{bmatrix} L'_1 & \text{on } E'_1 \\ \vdots & \vdots \\ L'_{n'} & \text{on } E'_{n'} \end{bmatrix}$$

and apply the above methods, it follows from *Mixture Monotonicity* that $a \succsim a'$ if and only if

$$\left[\sum_{i=1}^{n-1}(U(L_i)-U(L_{i+1}))\,\pi_i + U(L_n)\right] \geq \left[\sum_{j=1}^{n'-1}\left(U(L'_j)-U(L'_{j+1})\right)\pi'_j + U(L'_{n'})\right]$$

Hence, if we set, $\pi(\emptyset) := 0$, $\pi(S) := 1$ and $\pi\left(\cup_{j=1}^{i} E_i\right) := \pi_i$ then we have established that $\succsim$ can be represented by the functional

$$V\left(\left[\begin{array}{cc} L_1 & \text{on } E_1 \\ \vdots & \vdots \\ L_n & \text{on } E_n \end{array}\right]\right) = \sum_{i=1}^{n-1} \left(U(L_i) - U(L_{i+1})\right) \pi\left(\cup_{j=1}^{i} E_j\right) + U(L_n)$$

$$= U(L_1)\,\pi(E_1) + \sum_{i=2}^{n} \left(U(L_i)\right)\left(\pi\left(\cup_{j=1}^{i} E_j\right) - \pi\left(\cup_{j=1}^{i-1} E_j\right)\right).$$

Just remains to show that $\pi(.)$ to be additive.

That is, for any pair of events A and B,

$$\pi(A \cup B) = \pi(A) + \pi(B) - \pi(A \cap B).$$

To see that this indeed holds, consider

$$\frac{1}{2}\left[\begin{array}{cc} \delta_M & \text{on } A \cup B \\ \delta_m & \text{on } S - (A \cup B) \end{array}\right] + \frac{1}{2}\left[\begin{array}{cc} \delta_M & \text{on } A \cap B \\ \delta_m & \text{on } S - (A \cap B) \end{array}\right]$$

$$\sim \frac{1}{2}\left(\pi(A \cup B)\,\delta_M + (1 - \pi(A \cup B))\,\delta_m\right)$$

$$+\frac{1}{2}\left(\pi(A \cap B)\,\delta_M + (1 - \pi(A \cap B))\,\delta_m\right) \text{ (applying Independence twice)}$$

$$= \frac{1}{2}\left[\pi(A \cup B) + \pi(A \cap B)\right]\delta_M + \left(1 - \frac{\pi(A \cup B) + \pi(A \cap B)}{2}\right)\delta_m$$

But

$$\begin{aligned}
& \frac{1}{2}\left[\begin{array}{cc} \delta_M & \text{on } A\cup B \\ \delta_m & \text{on } S-(A\cup B) \end{array}\right] + \frac{1}{2}\left[\begin{array}{cc} \delta_M & \text{on } A\cap B \\ \delta_m & \text{on } S-(A\cap B) \end{array}\right] \\
= & \frac{1}{2}\left[\begin{array}{cc} \delta_M & \text{on } A \\ \delta_m & \text{on } B \\ \delta_m & \text{on } S-(A\cup B) \end{array}\right] + \frac{1}{2}\left[\begin{array}{cc} \delta_m & \text{on } A \\ \delta_M & \text{on } B \\ \delta_m & \text{on } S-(A\cup B) \end{array}\right] \\
\sim & \frac{1}{2}\left(\pi(A)\,\delta_M + (1-\pi(A))\,\delta_m\right) + \frac{1}{2}\left(\pi(B)\,\delta_M + (1-\pi(B))\,\delta_m\right) \\
= & \frac{1}{2}\left(\pi(A)+\pi(B)\right)\delta_M + \frac{1}{2}\left(1-\pi(A)-\pi(B)\right)\delta_m
\end{aligned}$$

So by *Mixture Monotonicity* it follows

$\pi(A\cup B)+\pi(A\cap B)=\pi(A)+\pi(B)$, as required.

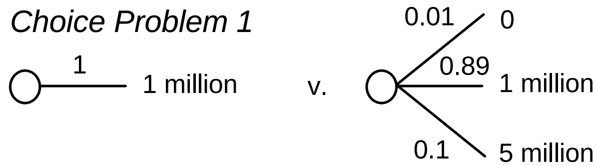
Hence we have established that $\succsim$ can be represented by the functional

$$\begin{aligned}
& V\left(\left[\begin{array}{cc} L_1 & \text{on } E_1 \\ \vdots & \vdots \\ L_n & \text{on } E_n \end{array}\right]\right) \\
= & U(L_1)\,\pi(E_1) + \sum_{i=2}^{n} \left(U(L_i)\right)\left(\pi\left(\cup_{j=1}^{i} E_j\right) - \pi\left(\cup_{j=1}^{i-1} E_j\right)\right) \\
= & \pi(E_1)\,U(L_1) + \pi(E_2)\,U(L_2) + \ldots + \pi(E_n)\,U(L_n).
\end{aligned}$$

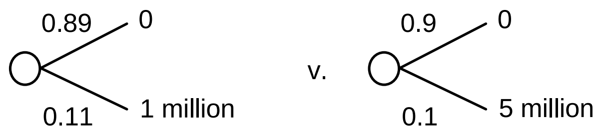
Utility Paradoxes and Rationality.

1. Violations of the 'Expected Utility Rule'

Choice Problem 1



Choice Problem 2



$$u(1) > 0.01u(0) + 0.89u(1) + 0.1u(5)$$

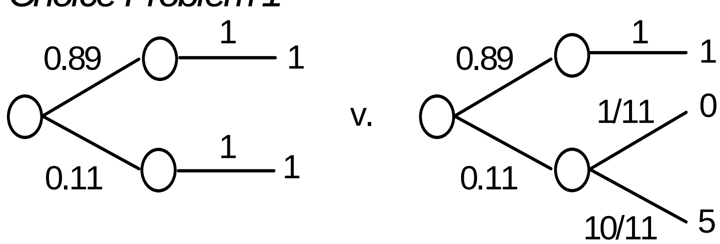
$$\Rightarrow u(1) > \frac{1}{11}u(0) + \frac{10}{11}u(5)$$

and

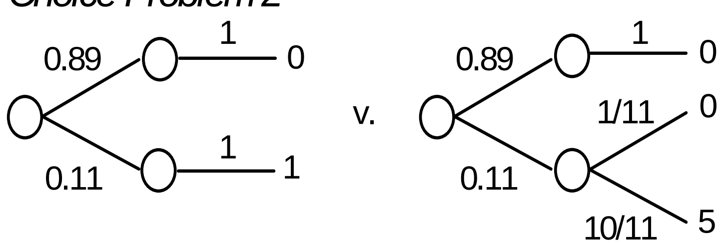
$$0.89u(0) + 0.11u(1) < 0.9u(0) + 0.1u(5)$$

$$\Rightarrow u(1) < \frac{1}{11}u(0) + \frac{10}{11}u(5)$$

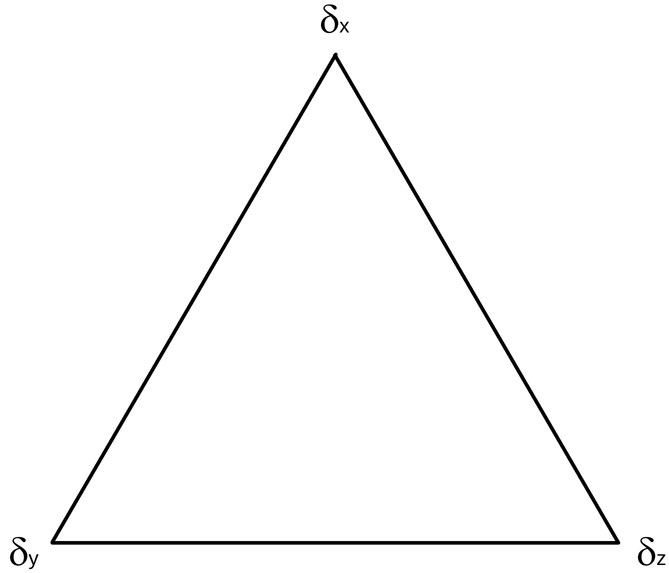
Choice Problem 1'



Choice Problem 2'



Are preferences linear in probabilities?



Can Beliefs Always be Represented by Probabilities?

Urn contains 90 balls. 30 are red. The other sixty are black and white balls in *unknown* proportions.

	Acts				
EVENT	a	b		a'	b'
Red	100	0		100	0
White	0	100		0	100
Black	0	0		100	100

$a \succ b \Rightarrow$

$$\begin{aligned} & \pi\,(\text{Red})\,U\,(\delta_{100}) + (1 - \pi\,(\text{Red}))\,U\,(\delta_0) \\ > \; & \pi\,(\text{White})\,U\,(\delta_{100}) + (1 - \pi\,(\text{White}))\,U\,(\delta_0) \end{aligned}$$

I.e. $\pi\,(\text{Red}) > \pi\,(\text{White})$

Can Beliefs Always be Represented by Probabilities?

Urn contains 90 balls. 30 are red. The other sixty are black and white balls in *unknown* proportions.

	Acts				
EVENT	a	b		a'	b'
Red	100	0		100	0
White	0	100		0	100
Black	0	0		100	100

$b' \succ a' \Rightarrow$

$$\begin{aligned} & \pi\,(\text{White or Black})\,U\,(\delta_{100}) + (1 - \pi\,(\text{White or Black}))\,U\,(\delta_0) \\ > \quad & \pi\,(\text{Red or Black})\,U\,(\delta_{100}) + (1 - \pi\,(\text{Red or Black}))\,U\,(\delta_0) \end{aligned}$$

I.e. $\pi\,(\text{Red}) < \pi\,(\text{White})$

Multiple Priors Expected Utility

Fix $\mathcal{D} \subset \Delta$. For any pair of acts

$$a = [L_1, E_1; \ldots; L_n, E_n] \text{ and } a' = [L'_1, E'_1; \ldots; L_{n'}, E_{n'}]$$

$a \succsim a' \Leftrightarrow$

$$\min_{\pi \in \mathcal{D}} \left(\sum_{i=1}^{n} \pi\,(E_i)\,U\,(L_i) \right) \geq \min_{\pi \in \mathcal{D}} \left(\sum_{j=1}^{n'} \pi\,(E'_j)\,U\,(L'_j) \right).$$

For example with

$$\mathcal{D} = \{(\pi_R, \pi_B, \pi_W) \in \Delta : \pi_R = 1/3,\ \pi_W \in \{1/6, 1/3, 1/2\}\}$$

$$\begin{aligned} V\,(a) &= 1/3 > 1/6 = V\,(b) \\ V\,(b') &= 2/3 > 1/2 = 1/3 + 1/6 = V\,(a') \end{aligned}$$

Role of Expected Utility in Economics

- Economists use EU as an *element* of models of choice in uncertain environments
  - asset allocation, insurance, saving, investment
  - any alternative needs to be capable of being incorporated into more complicated models of market behavior.
- All models are approximations. Challenge is to show departures from maximizing EU have consequences that are not minor for issues under examination.
  - e.g. departures from EU that are not systematic may not bias predictions of *market* behavior or outcomes.

- Growing body of literature showing some key features of insurance and financial markets can be better explained by models that allow for systematic departures from EU.
  - But always easier to obtain better fit by adding parameters to model.
  - Unclear, whether *out of sample* performance is superior to simpler models based on EU.
- For rest of course, we will assume DMs are *subjective expected utility maximizers*. Likely to be better approximation to real world behavior:
  - the more situation is a repeated event
  - the more significant the choice is for individual wealth
  - the more alternative gambles under consideration can be viewed as "local deviations" from each other.