ECON501 - Vector Differentiation

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October 2005

Abstract

Notes on vector differentiation and some simple economic applications and examples.

1. Functions of One Variable

$$g : \mathbb{R} \to \mathbb{R}$$

$$\text{derivative (slope) } \frac{d}{dx} g(\overline{x}) \equiv g'(\overline{x}) \equiv \lim_{h \to 0} \frac{g(\overline{x}+h) - g(\overline{x})}{h}$$

2nd derivative (rate of change of slope)

$$\frac{d^2}{dx^2} g(\overline{x}) \equiv g''(\overline{x}) \equiv \lim_{h \to 0} \frac{g'(\overline{x}+h) - g'(\overline{x})}{h}$$

2. Functions of Several Variables

2.1. Real-valued function of a vector x $(N \times 1)$[1]

$$g : \mathbb{R}^N \to \mathbb{R}$$

Partial derivative of g wrt x_n is defined as

$$\frac{\partial}{\partial x_n} g(\overline{x}) \equiv g_n(\overline{x}) \equiv \lim_{h \to 0} \frac{g(\overline{x}_1, \ldots, \overline{x}_n + h, \ldots, x_N) - g(\overline{x})}{h} \tag{1}$$

$$Dg(\overline{x}) \equiv D_x g(\overline{x}) \equiv [\nabla g(\overline{x})]^T \equiv \left[\frac{\partial}{\partial x_1} g(\overline{x}), \ldots, \frac{\partial}{\partial x_N} g(\overline{x}) \right] \tag{2}$$

I.e. the *gradient* of g evaluated at $x = \overline{x}$, and denoted $\nabla g(\overline{x})$, is the vector (i.e $N \times 1$ matrix) of partial derivatives.

[1] Vectors are assumed to be column vectors, i.e. $(N \times 1)$ matrices. Row vectors [$(1 \times N)$ matrices] will be denoted by vector transposed. E.g. $(x_1, \ldots, x_N) = x^T$

Example 1 *Let $u : \mathbb{R}^L_+ \to \mathbb{R}$ be a utility function for consumption bundles that can be represented by points in $\mathbb{R}^L_+$. $\nabla u(\overline{x})$ is then the gradient of the utility function at consumption bundle $\overline{x}$.*

$$I.e.\ \nabla u(\overline{x}) \equiv [Du(\overline{x})]^T \equiv \begin{bmatrix} \frac{\partial}{\partial x_1} u(\overline{x}) \\ \vdots \\ \frac{\partial}{\partial x_L} u(\overline{x}) \end{bmatrix}$$

2.2. Vector function of a vector x

$$g : \mathbb{R}^N \to \mathbb{R}^M \text{ I.e. } g \text{ maps points from } \mathbb{R}^N \text{ to points in } \mathbb{R}^M$$

Example 2 *Demand function $x(p, w)$ which will later be referred to as a* system of demand functions*, maps a price vector and income/wealth level to a consumption bundle. Denote $x_\ell(\overline{p}, \overline{w})$ as the quantity of good ℓ demanded by a consumer facing an L-dimensional vector of prices $\overline{p}$ and who has wealth level $\overline{w}$.*

$$I.e.\ x : \mathbb{R}^{L+1}_+ \to \mathbb{R}^L_+ \text{ and for each } \ell,\ x_\ell : \mathbb{R}^{L+1}_+ \to \mathbb{R}_+$$

Example 3 *The gradient of the utility function from example 1, may be viewed as a function from $\mathbb{R}^N$ into $\mathbb{R}^N$.*

The vector function $g : \mathbb{R}^N \to \mathbb{R}^M$ may be viewed as M scalar functions:

$$g_m : \mathbb{R}^N \to \mathbb{R},\ m = 1, \ldots, M.$$

From equation (2) we have

$$Dg_m(\overline{x}) \equiv \left[\frac{\partial}{\partial x_1} g_m(\overline{x}), \ldots, \frac{\partial}{\partial x_N} g_m(\overline{x})\right]$$

thus we may view the derivative of $g(x)$, evaluated at $x = \overline{x}$, as the M stacked row vectors of partial derivatives for each $g_m(\overline{x})$. Thus we have an $M \times N$ matrix of partial derivatives:

$$Dg(\overline{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} g_1(\overline{x}) & \ldots & \frac{\partial}{\partial x_N} g_1(\overline{x}) \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1} g_M(\overline{x}) & \ldots & \frac{\partial}{\partial x_N} g_M(\overline{x}) \end{bmatrix} \tag{3}$$

Notice that if we took the derivative of $g(x)$ at $\overline{x}$ w.r.t. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, holding $\begin{pmatrix} x_3 \\ \vdots \\ x_N \end{pmatrix}$ fixed at $\begin{pmatrix} \overline{x}_3 \\ \vdots \\ \overline{x}_N \end{pmatrix}$, our derivative would then be the *first two* columns of $Dg(\overline{x})$.

$$D_{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} g(\overline{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} g_1(\overline{x}) & \frac{\partial}{\partial x_2} g_1(\overline{x}) \\ \vdots & \vdots \\ \frac{\partial}{\partial x_1} g_M(\overline{x}) & \frac{\partial}{\partial x_2} g_M(\overline{x}) \end{bmatrix} \tag{4}$$

Example 4 *(Example 2 cont.): For our system of demand functions $x(p, w)$*

$$\begin{aligned} D_{\begin{pmatrix} p \\ w \end{pmatrix}} x(\overline{p}, \overline{w}) &= \begin{bmatrix} \frac{\partial}{\partial p_1} x_1(\overline{p}, \overline{w}) & \ldots & \frac{\partial}{\partial p_L} x_1(\overline{p}, \overline{w}) & \frac{\partial}{\partial w} x_1(\overline{p}, \overline{w}) \\ \vdots & & \vdots & \vdots \\ \frac{\partial}{\partial p_1} x_L(\overline{p}, \overline{w}) & \ldots & \frac{\partial}{\partial p_L} x_L(\overline{p}, \overline{w}) & \frac{\partial}{\partial w} x_L(\overline{p}, \overline{w}) \end{bmatrix} \\ &= \left[D_p x(\overline{p}, \overline{w}) \quad D_w(\overline{p}, \overline{w}) \right] \end{aligned} \tag{5}$$

where $D_p x(\overline{p}, \overline{w})$ is the $L \times L$ matrix of price partial derivatives and $D_w x(\overline{p}, \overline{w})$ is the $L \times 1$ matrix of wealth effects.

Example 5 *(Example 3 cont.): $\nabla u(\overline{x}) \equiv [Du(\overline{x})]^T$ is a function from $\mathbb{R}^L$ to $\mathbb{R}^L$. The derivative of this function (i.e. the "derivative* *of the gradient" of u) is called the Hessian of u.*

$$\begin{aligned}
H_u &\equiv \nabla^2 u(\overline{x}) \equiv D[\nabla u(\overline{x})] \equiv D^2 u(\overline{x}) \\
&\equiv \begin{bmatrix} \frac{\partial}{\partial x_1} u_1(\overline{x}) & \dots & \frac{\partial}{\partial x_L} u_1(\overline{x}) \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1} u_L(\overline{x}) & \dots & \frac{\partial}{\partial x_L} u_L(\overline{x}) \end{bmatrix} \\
&\equiv \begin{bmatrix} \frac{\partial^2}{\partial x_1 \partial x_1} u(\overline{x}) & \dots & \frac{\partial^2}{\partial x_1 \partial x_L} u(\overline{x}) \\ \vdots & & \vdots \\ \frac{\partial^2}{\partial x_L \partial x_1} u(\overline{x}) & \dots & \frac{\partial^2}{\partial x_L \partial x_L} u(\overline{x}) \end{bmatrix}
\end{aligned}$$

3. Applications

3.1. Differentiation of an Inner Product.

$$g(x) = p^T x = \sum_{\ell=1}^{L} p_\ell x_\ell$$

By equation (2)

$$Dg(\overline{x}) \equiv \left[\frac{\partial}{\partial x_1} g(\overline{x}), \dots, \frac{\partial}{\partial x_L} g(\overline{x})\right] = (p_1, \dots, p_L).$$

Notice that the budget hyperplane, $\{x \in \mathbb{R}^L | p^T x = w\}$, is the *level* set of the *linear* functon $g(x) = p^T x$. Thus the gradient of the budget hyperplane is the price vector p.

3.2. Euler's Theorem Applied to a Demand System.

$$x(\alpha p, \alpha w) \equiv x(p, w) \text{ for all } p, w \text{ and } \alpha > 0 \tag{6}$$

Differentiating equation (6) wrt α and evaluating at $\alpha = 1$, $p = \overline{p}$ and $w = \overline{w}$, from equation (5) we obtain

$$[D_p x(\overline{p}, \overline{w}) \quad D_w(\overline{p}, \overline{w})] \begin{pmatrix} \overline{p} \\ \overline{w} \end{pmatrix} = 0$$

and multiplying out

$$D_p x(\overline{p},\overline{w})\,\overline{p} + D_w(\overline{p},\overline{w})\,\overline{w} = 0 \tag{7}$$

To see this, let us focus on the demand for commodity 1. From (6) we have:-

$$x_1(\alpha\overline{p}_1, \alpha\overline{p}_2, \ldots, \alpha\overline{p}_L, \alpha w) \equiv x_1(\overline{p},\overline{w}) \tag{8}$$

When α changes, each price changes and income changes in the LHS. So we should expect $(L+1)$ partial derivatives appearing in our LHS expression for the effect on the demand for commodity 1 from a change in α.

Notice that the rate at which each price p_ℓ changes as α changes is $\overline{p}_n$ (i.e. $\frac{\partial}{\partial\alpha}(\alpha\overline{p}_\ell) = \overline{p}_\ell$) and the rate at which the demand for commodity 1 changes as p_ℓ changes is $\frac{\partial}{\partial p_\ell}x_1(\overline{p},\overline{w})$. So the effect on the quantity demanded of commodity 1 of a change in α *via* its effect on the price p_ℓ is $\frac{\partial}{\partial p_\ell}x_1(\overline{p},\overline{w}) \times \overline{p}_\ell$. Similarly the effect on the quantity demanded of commodity 1 from a change in α via its effect on wealth is $\frac{\partial}{\partial w}x_1(\overline{p},\overline{w}) \times \overline{w}$. So differentiating the LHS of equation (8) yields

$$\sum_{\ell=1}^{L} \frac{\partial}{\partial p_\ell} x_1(\overline{p},\overline{w}) \times \overline{p}_\ell + \frac{\partial}{\partial w} x_1(\overline{p},\overline{w}) \times \overline{w}$$

and as α does not appear on the RHS of equation (8), the derivative of the RHS wrt α is simply 0. Stacking the L equations, one for each commodity, we obtain (7), the result in matrix form.

Exercise 1 *Using the definitions for $D_p x(\overline{p},\overline{w})$ and $D_w(\overline{p},\overline{w})$, multiply out the matrices in (7) and check that you indeed obtain L equations of the form:-*

$$\sum_{k=1}^{L} \frac{\partial}{\partial p_k} x_\ell(\overline{p},\overline{w}) \times \overline{p}_k + \frac{\partial}{\partial w} x_\ell(\overline{p},\overline{w}) \times \overline{w} = 0 \tag{9}$$

Define

$$\varepsilon_{\ell k} \equiv \frac{\partial}{\partial p_k} x_\ell (\overline{p}, \overline{w}) \times \frac{\overline{p}_k}{x_\ell (\overline{p}, \overline{w})}$$

as the price elasticity of demand for commodity ℓ wrt price p_k; and

$$\varepsilon_{\ell w} \equiv \frac{\partial}{\partial w} x_\ell (\overline{p}, \overline{w}) \times \frac{\overline{w}}{x_\ell (\overline{p}, \overline{w})}$$

as the wealth elasticity of demand for commodity ℓ.

Dividing equation (9) by $x_\ell (\overline{p}, \overline{w})$ we obtain:-

$$\sum_{k=1}^{L} \varepsilon_{\ell k} + \varepsilon_{\ell w} = 0 \tag{10}$$

I.e. the sum of the price elasticities of demand for commodity ℓ is equal to minus its wealth elasticity.

3.3. Differentiating Walras' Law

$$p^T x (p, w) \equiv w \tag{11}$$

3.3.1. Differentiating with respect to wealth.

If we differentiate (11) wrt w and evaluate at $p = \overline{p}$ and $w = \overline{w}$ we obtain

$$\begin{aligned} D_w \left[\overline{p}^T x (\overline{p}, \overline{w}) \right] = \overline{p}^T D_w x (\overline{p}, \overline{w}) &= 1 \\ \text{or } \sum_{\ell=1}^{L} \overline{p}_\ell \times \frac{\partial}{\partial w} x_\ell (\overline{p}, \overline{w}) &= 1 \end{aligned} \tag{12}$$

Equation (12) is known as the *Engel Aggregation*. If we let

$$m_\ell \equiv \overline{p}_\ell \times \frac{\partial}{\partial w} x_\ell (\overline{p}, \overline{w})$$

denote the *marginal propensity spend* on commodity ℓ, then the Engel aggregation may be interpreted as saying (loosely):

> "*The sum of the marginal propensities to spend out of a dollar on all commodities equals a dollar.*"

3.3.2. Differentiating with respect to prices.

If we differentiate (11) wrt p and evaluate at $p = \overline{p}$ and $w = \overline{w}$ we obtain

$$D_p \left[\overline{p}^T x \left(\overline{p}, \overline{w} \right) \right] = \overline{p}^T D_p x \left(\overline{p}, \overline{w} \right) + \left[x \left(\overline{p}, \overline{w} \right) \right]^T = 0 \tag{13}$$

Equation (13) is known as the *Cournot Aggregation.* If we multiply out the matrices in (13) we obtain L equations fo the form:-

$$\sum_{k=1}^{L} \overline{p}_k \times \frac{\partial}{\partial p_\ell} x_k \left(\overline{p}, \overline{w} \right) + x_\ell \left(\overline{p}, \overline{w} \right) = 0 \tag{14}$$

In words it says that the sum of the rate of change on expenditure of all commodities for an increase in the price p_ℓ is equal to minus the quantity of commodity ℓ currently demanded. That is, if p_ℓ rises, the rate at which wealth would have to rise to enable the consumer to purchase the original consumption bundle is $x_\ell \left(\overline{p}, \overline{w} \right)$. In other words the *purchasing power* of the consumer falls at the rate $x_\ell \left(\overline{p}, \overline{w} \right)$ as the price of that commodity rises.