**Rice University**

*Fall Semester Final Examination 2007*

**ECON501 Advanced Microeconomic Theory**

*Writing Period:* **Three Hours**

*Permitted Materials: English/Foreign Language Dictionaries and non-programmable calculators*

*There are five pages (including this title page) and five questions. You should attempt all parts of all five questions. The total points for the exam is one hundred and eighty (180).*

1. **[60 Points]**

Consider a preference relation that can be represented by the utility fuction

$$u(x_1, x_2, x_3, x_4) = \left(\frac{x_1}{\alpha}\right)^{\alpha}\left(\frac{x_2}{1-\alpha}\right)^{1-\alpha} + 2\sqrt{\left(\frac{x_3}{\beta}\right)^{\beta}\left(\frac{x_4}{1-\beta}\right)^{1-\beta}}.$$

(a) Derive the uncompensated demand function for the region of the parameter space in which the consumer is choosing positive amounts of all four commodities. Evaluate the income elasticities of demand for each of the four commodities.

*Short cut* $u(x_1, x_2, x_3, x_4) = u_A(x_1, x_2) + u_B(x_3, x_4)$, *so if* $e_A = p_1x_1 + p_2x_2$ *and* $e_B = p_3x_3 + p_4x_4$, *then problem can be reexpressed as*

$$\begin{aligned}
&\max_{\langle e_A, e_B\rangle} v_A(p_1, p_2, e_A) + v_B(p_3, p_4, e_B) \text{ s.t. } e_A + e_B \leq w \\
&\textit{where } v_A(p_1, p_2, e_A) = \max_{\langle x_1, x_2\rangle} u_A(x_1, x_2) \text{ s.t. } p_1x_1 + p_2x_2 \leq e_A \\
&\textit{and } v_B(p_3, p_4, e_B) = \max_{\langle x_3, x_4\rangle} u_B(x_3, x_4) \text{ s.t. } p_3x_3 + p_4x_4 \leq e_B
\end{aligned}$$

*Since* $u_A(x_1, x_2)$ *and* $u_B(x_3, x_4)$ *are monotonic transformations of Cobb-Douglas utilities, we have*

$$\begin{aligned}
x_1(p_1, p_2, e_A) &= \frac{\alpha e_A}{p_1},\ x_1(p_1, p_2, e_A) = \frac{(1-\alpha)e_A}{p_2}, \\
x_3(p_3, p_4, e_B) &= \frac{\beta e_B}{p_3},\ x_4(p_3, p_4, e_B) = \frac{(1-\beta)e_B}{p_4}.
\end{aligned}$$

*Hence*

$$v_A(p_1, p_2, e_A) = \frac{e_A}{p_1^{\alpha}p_2^{(1-\alpha)}} \textit{ and } v_B(p_1, p_2, e_A) = 2\sqrt{\frac{e_B}{p_3^{\beta}p_4^{(1-\beta)}}}$$

*So the expenditure between groups problem can be written as*

$$\max_{e_B} \frac{w - e_B}{p_1^{\alpha}p_2^{(1-\alpha)}} + 2\sqrt{\frac{e_B}{p_3^{\beta}p_4^{(1-\beta)}}}$$

*FONC*

$$\frac{-1}{p_1^{\alpha}p_2^{(1-\alpha)}} + \sqrt{\frac{1}{e_B^*\left[p_3^{\beta}p_4^{(1-\beta)}\right]}} = 0, \textit{ since } e_B > 0.$$

*Yielding*

$$e_B^* = \frac{p_1^{2\alpha}p_2^{2(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}} \textit{ and } e_A^* = w - \frac{p_1^{2\alpha}p_2^{2(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}}$$

*Substituting back yields the uncompensated demand*

$$\begin{aligned}
x_1(p_1,p_2,p_3,p_4,w) &= \frac{\alpha}{p_1}\left(w - \frac{p_1^{2\alpha}p_2^{2(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}}\right) \\
x_2(p_1,p_2,p_3,p_4,w) &= \frac{(1-\alpha)}{p_2}\left(w - \frac{p_1^{2\alpha}p_2^{2(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}}\right) \\
x_3(p_1,p_2,p_3,p_4,w) &= \frac{\beta}{p_3} \times \frac{p_1^{2\alpha}p_2^{2(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}} \\
x_4(p_1,p_2,p_3,p_4,w) &= \frac{(1-\beta)}{p_4} \times \frac{p_1^{2\alpha}p_2^{2(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}}
\end{aligned}$$

*Computing the marginal propensities to consume out of wealth:*

$$p_1\frac{\partial x_1}{\partial w} = \alpha,\ p_2\frac{\partial x_2}{\partial w} = (1-\alpha),\ p_3\frac{\partial x_3}{\partial w} = 0,\ p_4\frac{\partial x_4}{\partial w} = 0.$$

(b) Using your answer from ($a$) derive the indirect utility function.
*From (a) we have*

$$\begin{aligned}
v(p_1,p_2,p_3,w) &= \frac{e_A^*}{p_1^{\alpha}p_2^{(1-\alpha)}} + 2\sqrt{\frac{e_B^*}{p_3^{\beta}p_4^{(1-\beta)}}} \\
&= \frac{w}{p_1^{\alpha}p_2^{(1-\alpha)}} - \frac{p_1^{\alpha}p_2^{(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}} + 2\frac{p_1^{\alpha}p_2^{(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}} \\
&= \frac{w}{p_1^{\alpha}p_2^{(1-\alpha)}} + \frac{p_1^{\alpha}p_2^{(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}}.
\end{aligned}$$

(c) Derive the compensated demand function.
*From the fundamental identity*

$$v(p_1,p_2,p_3,e(p_1,p_2,p_3,p_4,u)) = u$$

*we have*

$$\frac{e(p_1,p_2,p_3,p_4,u)}{p_1^{\alpha}p_2^{(1-\alpha)}} + \frac{p_1^{\alpha}p_2^{(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}} = u$$

*and hence*

$$e(p_1,p_2,p_3,p_4,u) = up_1^{\alpha}p_2^{(1-\alpha)} - \frac{p_1^{2\alpha}p_2^{2(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}}$$

*By differentiating the expenditure function we obtain the compensated demands*

*(Shepherd's lemma).*

$$\begin{aligned}
h_1(p_1,p_2,p_3,p_4,u) &= \frac{\alpha}{p_1}\left[e(p_1,p_2,p_3,p_4,u) - \frac{p_1^{2\alpha}p_2^{2(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}}\right]\\
h_2(p_1,p_2,p_3,p_4,u) &= \frac{(1-\alpha)}{p_2}\left[e(p_1,p_2,p_3,p_4,u) - \frac{p_1^{2\alpha}p_2^{2(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}}\right]\\
h_3(p_1,p_2,p_3,p_4,u) &= \frac{\beta}{p_3}\times\frac{p_1^{2\alpha}p_2^{2(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}}\\
h_4(p_1,p_2,p_3,p_4,u) &= \frac{(1-\beta)}{p_4}\times\frac{p_1^{2\alpha}p_2^{2(1-\alpha)}}{p_3^{\beta}p_4^{(1-\beta)}}
\end{aligned}$$

Recall in lectures that the equivalent variation of a change in prices and income from $(p^0,w^0)$ to $(p^1,w^1)$ was defined to be

$$\mathrm{EV} = e\left(p^0, v\left(p^1,w^1\right)\right) - w^0.$$

Further recall that if **wealth is unaltered** and the change in prices is caused by the imposition of commodity taxes then the deadweight loss (DWL) or excess burden of the taxes is given by

$$\mathrm{DWL} = -\,\mathrm{EV} - \sum\nolimits_{\ell=1}^{L} t_\ell x_\ell\left(p^1,w^0\right).$$

**(d)** Briefly explain why this measure may be viewed as a deadweight loss to (social) economic efficiency.

**(e)** Suppose the initial budget constraint has $p^0=(1,1,1,1)$ and $w^0=36$. Suppose further that the consumer's preferences parameters are $\alpha=1/2$ and $\beta=1/3$. Calculate the DWL of the imposition of a specific tax of 3 on good 1, that leads to the price of good 1 rising to 4.

$$\mathrm{DWL} = w^0 - e\left(p^0, v\left(p^1,w^1\right)\right) - \sum\nolimits_{\ell=1}^{L} t_\ell x_\ell\left(p^1,w^0\right)$$

*From answers above, we have*

$$\begin{aligned}
v(4,1,1,1,36) &= \frac{w}{\sqrt{p_1p_2}} + \frac{\sqrt{p_1p_2}}{p_1^{1/3}p_2^{2/3}} = \frac{36}{2}+2 = 20,\\
e(1,1,1,1,20) &= u\sqrt{p_1p_2} - \frac{p_1p_2}{p_1^{1/3}p_2^{2/3}} = 20-1 = 19\\
x_1(4,1,1,1,36) &= \frac{1/2}{p_1}\left(w - \frac{p_1p_2}{p_1^{1/3}p_2^{2/3}}\right)\\
&= \frac{1}{8}(36-4) = 4
\end{aligned}$$

*Hence*

$$\mathrm{DWL} = 36 - 19 - 3\times 4 = 5$$

**(f)** Again starting from the initial position corresponding to $p^0 = (1, 1, 1, 1)$ and $w^0 = 36$ compute the DWL arising from the imposition of a specific tax of 3 on good 1, and specific taxes of 1 on both goods 3 and 4 resulting in the price of good 1 rising to 4 and the prices of goods 3 and 4 both rising to 2.

*From answers above, we have*

$$\begin{aligned}
v(4,1,2,2,36) &= \frac{w}{\sqrt{p_1p_2}} + \frac{\sqrt{p_1p_2}}{p_3^{1/3}p_4^{2/3}} = \frac{36}{2} + 1 = 19, \\
e(1,1,1,1,19) &= u\sqrt{p_1p_2} - \frac{p_1p_2}{p_3^{1/3}p_4^{2/3}} = 19 - 1 = 18, \\
x_1(4,1,2,2,36) &= \frac{1/2}{p_1}\left(w - \frac{p_1p_2}{p_3^{1/3}p_4^{2/3}}\right) \\
&= \frac{1}{8}(36-2) = \frac{34}{8} = \frac{17}{4} \\
x_3(4,1,2,2,36) &= \frac{1/3}{p_3} \times \frac{p_1p_2}{p_3^{1/3}p_4^{2/3}} = \frac{1}{3} \\
x_4(9,1,3,3,36) &= \frac{2/3}{p_4} \times \frac{p_1p_2}{p_3^{1/3}p_4^{2/3}} = \frac{2}{3}
\end{aligned}$$

*Hence*

$$\text{DWL} = 36 - 18 - 3 \times \frac{17}{4} - 1 \times \frac{1}{3} - 1 \times \frac{2}{3} = \frac{17}{4}$$

2. **[15 Points]** Consider the following four lotteries:

$$L_1(x) = \begin{cases} 1 & \text{if } x = \$3,000 \\ 0 & \text{otherwise} \end{cases}, \quad L_2(x) = \begin{cases} 0.8 & \text{if } x = \$4,000 \\ 0.2 & \text{if } x = \$0 \\ 0 & \text{otherwise} \end{cases}$$

$$L_3(x) = \begin{cases} 0.25 & \text{if } x = \$3,000 \\ 0.75 & \text{if } x = \$0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } L_4(x) = \begin{cases} 0.2 & \text{if } x = \$4,000 \\ 0.8 & \text{if } x = \$0 \\ 0 & \text{otherwise} \end{cases}$$

Show that if for the preference relation $\succcurlyeq$ we have

$$L_1 \succ L_2 \text{ and } L_4 \succ L_3$$

then $\succcurlyeq_{\mathcal{L}}$ does not admit an expected utility representation. Which of the axioms for subjective expected utility theory does this pattern of preferences violate? State this axiom and show how the this pattern of preference is inconsistent with the axiom.

*Wlog, set* $u(4,000) := 1$ *and* $u(0) := 0$. *Let* $u(3000) = u$.

$$\begin{aligned}
L_1 &\succ L_2 \Rightarrow u > 0.8 \times 1 + 0.2 \times 0 \Rightarrow u > 0.8 \\
L_4 &\succ L_3 \Rightarrow 0.2 \times 1 + 0.8 \times 0 > 0.25 \times u + 0.75 \times 0 \\
&\Rightarrow 0.2 > 0.25u \Rightarrow 0.8 > u, \text{ a contradiction!}
\end{aligned}$$

*Notice,*

$$L_3 = 0.25 \times L_1 + 0.75\delta_0 \text{ and } L_4 = 0.25 \times L_2 + 0.75\delta_0$$

*The independence axiom states, for any three lotteries $L$, $L'$ and $L''$ and any $\alpha$ in $(0,1]$ :*

$$L \succsim L' \Leftrightarrow \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$$

*One implication is that if $L \succ L'$ then $\alpha L + (1-\alpha) L'' \succ \alpha L' + (1-\alpha) L''$. But taking,*

$$L := L_1,\ L' := L_2,\ L'' := \delta_0 \textit{ and } \alpha = 0.25,$$

*we have*

$$L_1 \succ\ L_2 \textit{ but } 0.25 \times L_2 + 0.75\delta_0 \succ 0.25 \times L_1 + 0.75\delta_0,$$

*a violation of independence.*

3. [**15 Points**] An individual is an expected utility maximizer described by the intertemporally additive preference-scaling function

$$u(c_0) + \beta u(c_1)$$

where $u(\cdot)$ is a strictly concave function with $u'''(c) > 0$. The individual has current income $I_0$. The individual can buy bonds at unit price $p = \beta$ which pay out in the next period one unit of consumption per unit of bond held. Compare the individual's demand for bonds in the case where her future income is certain and equal to $I_0$, and the situation in which there is a fifty percent chance future income is $I_0 + \varepsilon$ and a fifty percent chance future income is $I_0 - \varepsilon$.

*Let $q$ denote the quantity of bonds the individual buys in period 0. Let $c_1$ be her consumption in period 1. Her maximization problem may be expressed as follows*

$$\max_q u(c_0) + \beta u(c_1)$$

$$\begin{aligned} s.t.\ c_0 &= I_0 - pq \\ c_1 &= I_0 q \end{aligned}$$

*or*

$$\max_q u(I_0 - pq) + \beta u(I_0 + q)$$

*FONCe*

$$q : -u'(I_0 - pq^*)\, p + \beta u'(I_0 + q^*) = 0$$

*Since $p = \beta$, and $u$ is strictly concave, it follows unique solution is $q^* = 0$.*

*Now let $c_1$ be her consumption in period 1 when her income is $I_0 + \varepsilon$ and let $c_2$ be her consumption in period 1 when her income is $I_0 - \varepsilon$. Her maximization problem may now be expressed as follows*

$$\max_q u(c_0) + \beta\left(\frac{1}{2}u(c_1) + \frac{1}{2}u(c_2)\right)$$

$$\begin{aligned} s.t.\ c_0 &= I_0 - pq \\ c_1 &= I_0 + \varepsilon + q \\ c_2 &= I_0 - \varepsilon + q \end{aligned}$$

*or*

$$\max_q u\left(I_0 - pq\right) + \beta\left(\frac{1}{2}u\left(I_0 + \varepsilon + q\right) + \frac{1}{2}u\left(I_0 - \varepsilon + q\right)\right)$$

*FONC*

$$q : -u'\left(I_0 - pq^{**}\right)p + \beta\left(\frac{1}{2}u'\left(I_0 + \varepsilon + q^{**}\right) + \frac{1}{2}u'\left(I_0 - \varepsilon + q^{**}\right)\right) = 0$$

*CLAIM* $q^{**} > q^* = 0$. *To see this, notice that LHS of FOC setting* $q = 0$ *yields*

$$\begin{aligned} & -u'\left(I_0\right)\beta + \beta\left(\frac{1}{2}u'\left(I_0 + \varepsilon\right) + \frac{1}{2}u'\left(I_0 - \varepsilon\right)\right) \\ > \ & \beta\left(-u'\left(I_0\right) + u'\left(\frac{1}{2}\left(I_0 + \varepsilon\right) + \frac{1}{2}\left(I_0 - \varepsilon\right)\right)\right) = 0 \end{aligned}$$

*The inequality follows from Jensen's inequality applied to a convex function ($u'$ is a convex function since $u''' > 0$). Thus we the individual's demand for bonds (i.e. for 'saving' ) is greater when she faces uncertain future income than when she faces certain future income. This comparative static effect (which obtains since $u'$ is convex) is an illustration of the 'precautionary motive for saving'.*

4. **[30 Points]** A price-taking firm produces output $q$ from inputs $z_1$ and $z_2$ according to a differentiable concave production function $f\left(z_1, z_2\right)$. The price of its output is $p > 0$ and the price of its inputs are $\left(w_1, w_2\right) \gg 0$. However, there are two unusual things about this firm. First, rather than maximizing profit, the firm maximizes revenue (the manager wants his firm to have bigger dollar sales than any other). Second, the firm is cash constrained. In particular, it has only $C$ dollars on hand before production and, as a result, its total expenditures on inputs cannot exceed $C$.

   Suppose one of your econometrician colleagues tells you that she has used repeated observations of the firm's revenue under various output prices, and levels of the financial constraint and has determined that the *log* of the firm's revenue level $R$ can be expressed as the following function of the variables $\left(p, w_1, w_2, C\right)$ :

   $$\ln R\left(p, w_1, w_2, C\right) = \ln p + \left[\gamma + \ln C - \alpha \ln w_1 - (1 - \alpha)\ln w_2\right]$$

   (a) What is the firm's use of inputs $z_1$ and $z_2$ and its output $q$ when prices are $\left(p_1, w_1, w_2\right)$ and it has $C$ dollars of cash on hand? (15 points).

   (b) Using your answer to ($a$) show the firm's production function is Cobb-Douglas. (15 points).

5. [**15 Points**] A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$, is called **superadditive** if $f\left(x^1 + x^2\right) \geq f\left(x^1\right) + f\left(x^2\right)$. Show that every cost function is superadditive in input prices. Use this to prove the cost function is non-decreasing in input prices without requiring it to be differentiable.

   *Fix output target $q > 0$. Let $z^1 \in \mathbb{R}_+^{L-1}$ (respectively, $z^2$, $z^3$) be the cost-minimizing input vector for the production of $q$ units of the output when the firm faces input prices $w^1 \in \mathbb{R}_+^{L-1}$ (respectively, $w^2$, $w^1 + w^2$) Since these input bundles are cost-minimizing for their respective input price vectors we have*

$$\begin{aligned} c\left(w^1, q\right) &= w^1 \cdot z^1 \leq w^1 \cdot z^3 \\ c\left(w^2, q\right) &= w^2 \cdot z^2 \leq w^2 \cdot z^3 \end{aligned}$$

   *Adding these two inequalities, yields*

$$c\left(w^1, q\right) + c\left(w^2, q\right) \leq \left(w^1 + w^2\right) \cdot z^3 = c\left(w^1 + w^2, q\right),$$

   *as required.*

   *To see that the cost function is non-decreasing in prices, take any two input price vectors, $w'$ and $w''$. If $w'' - w' = d \geq 0$, then by taking $w^1 := w'$, $w^2 := d$ (and hence $w^1 + w^2 = w''$, we have, by superadditivity of the cost function,*

$$c\left(w''\right) \geq c\left(w'\right) + c\left(d\right)$$

   *Hence*

$$c\left(w''\right) - c\left(w'\right) = c\left(d\right) \geq 0, \text{ as required.}$$

6. [**45 Points**] **Throughout this question, assume all functions mentioned are twice continuously differentiable.**

   In Ruritania, the electricity industry is perfectly competitive both in the input and output markets. Let $B_e(q)$ represent a schedule of benefits generated by the consumption of electricity, with $B_e(0) = 0$, $B_e'(q) > 0$ and $B_e''(q) < 0$. Production opportunities in this industry is summarized by the aggregate production function:

$$q = f\left(K_e, L, z\right),$$

   where $q$ is the (maximum) amount of electricity that can be generated when $K_e$ units of capital are employed in electricity generation, in conjunction with $L$ units of labor and $z$ units of coal. Suppose $f$ is strictly increasing in all its arguments and strictly concave and that capital and labor are elastically supplied at prices $r$ and $w$, respectively. Suppose further that coal is competitively supplied according to a technology embodied in the cost function $C(z)$ (that is, the cost of supplying $z$ units of coal to the market is $C(z)$, where $C(0) = 0$, $C' > 0$ and $C'' > 0$.)

   Using coal to produce electricity generates as sulphur as a by-product. Left untreated sulphur is a pollutant but it can be made valuable if delivered in a pure state. Suppose the by-production of sulphur from burning coal is a linear activity, so that without loss of generality we can choose units for sulphur to be such that using $z$ units of coal generates $z$ units of sulphur as by-product. Let

$$y = g(K_s, z)$$

represent the production of pure sulphur from this joint product with $K_s$ being the units of capital involved in the "purification" process (also available at [constant] unit price of $r$). The function $g$ is strictly increasing in all arguments, strictly concave, and

$$\lim_{K_s \to \infty} g(K_s, z) \leq z \text{ for all } z.$$

The benefit for purified sulphur is given by the schedule $B_s(y)$, with $B_s'(y) > 0$, $B_s''(y) < 0$ and $B(0) = 0$. Unpurified sulphur will be emitted as a pollutant into the environment. The amount of pollutant $d$ will be

$$d = z - g(K_s, z).$$

The social harm it will cause is given by the schedule $H(d)$, with $H' > 0$, $H'' > 0$ and $H(0) = 0$.

(a) Characterize the economic efficient outcome for this configuration. (10 points).

*To characterize the economically efficient (Pareto optimal) outcome, consider maximizing*

$$\max_{\langle K_e, K_s, L, z\rangle} B_e(q) + B_s(y) - rK_e - rK_e - wL - C(z) - H(d)$$

*subject to*

$$\begin{aligned} q &= f(K_e, L, z) && (1)\\ y &= g(K_s, z) && (2)\\ d &= z - g(K_s, z) && (3) \end{aligned}$$

*Hence maximization can be re-expressed as:*

$$\max_{\langle K_e, K_s, L, z\rangle} B_e(f(K_e, L, z)) + B(g(K_s, z)) - rK_z - rK_y - wL - C(z) - H(z - g(K_s, z))$$

*FONC:*

$$\begin{aligned} K_e &: B_e'(q^*) f_K(K_e^*, L^*, z^*) - r = 0 && (4)\\ K_s &: [B_s'(y^*) + H'(d^*)] g_K(K_s, z) - r = 0 && (5)\\ L &: B_e'(q^*) f_L(K_e^*, L^*, z^*) - r = 0 && (6)\\ z &: B_e'(q^*) f_z(K_e^*, L^*, z^*) + B_s'(y^*) g_z(K_s, z) - C'(z) - H'(d^*)(1 - g_z(K_s, z)) = 0 && (7) \end{aligned}$$

*Each has obvious interpretation with the possible exception of the last. There, however, the equality of the marginal benefit generated by employing the input is required to equal its full marginal social cost.*

(b) Show that efficiency can be achieved even if $H(d)$ were ignored by producers in a competitive market if the government imposed an appropriate emission charge on $d$. (15 points).

*It is perhaps a miracle of modern economic theory that a single effluent charge can be shown to guarantee that the social optimum just characterized in (a) will be produced by a competitive economy. To see how, let $t$ be the emission charge on $d$, so that maximzing*

$$B_e(q) + B_s(y) - rK_e - rK_e - wL - C(z) - td$$

*with respect to $K_e$, $K_s$, $L$ and $z$ given the conditions recorded in equations* (2) *through* (3) *models the competitive economy's equilibrium behavior (by the first fundamental welfare theorem for partial equlibrium). The outcome, indicated by that notation is then the solution to a new set of first-order conditions:*

$$K_e \ : \ B_e'(\hat{q}) f_K\left(\hat{K}_e, \hat{L}, \hat{z}\right) - r = 0 \tag{8}$$

$$K_s \ : \ [B_s'(\hat{y}) + t]\, g_K\left(\hat{K}_s, \hat{z}\right) - r = 0 \tag{9}$$

$$L \ : \ B_e'(\hat{q}) f_L\left(\hat{K}_e, \hat{L}, \hat{z}\right) - r = 0 \tag{10}$$

$$z \ : \ B_e'(\hat{q}) f_z\left(\hat{K}_e, \hat{L}, \hat{z}\right) + B_s'(\hat{y})\, g_z\left(\hat{K}_s, \hat{z}\right) - C'(\hat{z}) - t\left(1 - g_z\left(\hat{K}_s, \hat{z}\right)\right) \tag{11}$$

*.Setting $t = H'(d^*)$, however, means equations* (4) *through* (7) *duplicate equations* (8) *through* (11). *As a result,*

$$\hat{K}_e = K_e^*,\ \hat{K}_s = K_e^*,\ \hat{L} = L^*,\ \hat{z} = z^*,\ \hat{q} = q^*,\ \hat{y} = y^* \text{ and } \hat{d} = d^*.$$

(c) Show that a system that charged a tax $t$ on the use of coal and paid a subsidy $s$ for the production of purified sulphur could also achieve optimality. (15 points).

*Giving a subsidy $s$ for the sale of purified sulphur and a tax $t$ charged on the use of coal as an input, the competitive solution maximizes*

$$B_e(q) + B_s(y) - rK_e - rK_e - wL - C(z) - tz + sy$$

*with respect to $K_e$, $K_s$, $L$ and $z$, equations* (2) *through* (3) *still hold, and the first-order conditions characterize a third vector of solutions:*

$$K_e \ : \ B_e'(\widetilde{q}) f_K\left(\widetilde{K}_e, \widetilde{L}, \widetilde{z}\right) - r = 0 \tag{12}$$

$$K_s \ : \ [B_s'(\widetilde{y}) + s]\, g_K\left(\widetilde{K}_s, \widetilde{z}\right) - r = 0 \tag{13}$$

$$L \ : \ B_e'(\widetilde{q}) f_L\left(\widetilde{K}_e, \widetilde{L}, \widetilde{z}\right) - r = 0 \tag{14}$$

$$z \ : \ B_e'(\widetilde{q}) f_z\left(\widetilde{K}_e, \widetilde{L}, \widetilde{z}\right) + B_s'(\hat{y})\, g_z\left(\widetilde{K}_s, \widetilde{z}\right) - C'(\hat{z}) + s g_z\left(\widetilde{K}_s, \widetilde{z}\right) - t = 0 \tag{15}$$

*Setting $s = t = H'(d^*)$, these equations duplicate equations* (8) *through* (11) *leading again to the efficient outcome.*

(d) Contrast the informational requirements of these alternative systems (i.e., the regulative structures embodied in parts ($b$) and ($c$). (5 points).

*Comparing the information required to enact the policies envisioned in parts (b) and (c), notice both require some notion of the optimal, total level of emissions so that* $H'(d^*)$ *can be evaluated. For an effluent tax to work, moreover, individual emission levels must be observed so that the required individual pollution fees can be collected. Therein lies a significant informational problem, because effluent is a difficult commodity to monitor effectively. On the other hand, the subsidy/input charge mechanism requires information about the quantity of material inputs being employed (more easily measured than effluent, perhaps at point of supply) and the quantity of useable, cleansed pollutant being sold (again an easier measurement).*