**Rice University**

*Fall Semester Final Examination 2006*

**ECON501 Advanced Microeconomic Theory**

*Writing Period:* **Three Hours**

*Permitted Materials: English/Foreign Language Dictionaries and non-programmable calculators*

*You should attempt all questions. The total points for the exam is one hundred and eighty (180).*

1. **[90 Points]**

   **(a)** Solve the utility maximization problem and derive the indirect utility function for preferences represented by the utility function

$$u(x_1, x_2) = \ln x_1 + x_2$$

Verify the indirect utility function satisfies all the requisite properties. (25 points).

**ANS:** *Set up the Lagrangean,*

$$\mathcal{L} = \ln x_1 + x_2 - \lambda(p_1 x_1 - p_2 x_2 - W)$$

*First order necessary conditions*

$$\begin{array}{rcc} x_1 : & x_1^{-1} - \lambda p_1 \leq 0 & (= 0 \text{ if } x_1 > 0) \\ x_2 : & 1 - \lambda p_2 \leq 0 & (= 0 \text{ if } x_2 > 0) \\ \lambda : & p_1 x_1 - p_2 x_2 - W \leq 0 & (= 0 \text{ if } \lambda > 0) \end{array}$$

*From FONC we see $x_1 > 0$ and since preferences are $\ell ns$ it also follows that the budget constraint holds with equality (i.e. $\lambda > 0$.) Hence solving the FONCs yields*

$$\begin{aligned} x_1(p_1, p_2, W) &= \begin{cases} p_2/p_1 & \text{if } W \geq p_2 \\ W/p_1 & \text{if } W < p_2 \end{cases} \\ &= \frac{\min(p_2, W)}{p_1} \\ x_2(p_1, p_2, W) &= \begin{cases} (W - p_2)/p_2 & \text{if } W \geq p_2 \\ 0 & \text{if } W < p_2 \end{cases} \\ &= \frac{\max(W - p_2, 0)}{p_2} \end{aligned}$$

*By plugging the solution into the utility function we obtain the indirect utility function:*

$$\begin{aligned} V(p_1, p_2, W) &= \ln x_1(p_1, p_2, W) + x_2(p_1, p_2, W) \\ &= \ln[\min(p_2, W)] - \ln p_1 + \frac{\max(W - p_2, 0)}{p_2} \end{aligned}$$

*We have*

$$\begin{aligned} \frac{\partial V}{\partial W} &= \begin{cases} W^{-1} & \text{if } W < p_2 \\ p_2^{-1} & \text{if } W \geq p_2 \end{cases} > 0 \text{ i.e. strictly increasing in wealth} \\ \frac{\partial V}{\partial p_1} &= -\frac{1}{p_1} \leq 0 \text{ i.e. non-decreasing in } p_1 \\ \frac{\partial V}{\partial p_2} &= \begin{cases} 0 & \text{if } W < p_2 \\ -(W - p_2)/p_2 & \text{if } W \geq p_2 \end{cases} \leq 0 \text{ i.e. non-decreasing in } p_2 \end{aligned}$$

*In the region* $W < p_2$

$$V(p_1, p_2, W) = \ln(W/p_1)$$

*And this is quasiconvex since if*

$$\begin{aligned} V(p_1, p_2, W) &\leq v \Rightarrow W \leq p_1 \exp(v) \Rightarrow \alpha W \leq \alpha p_1 \exp(v) \\ V(p_1', p_2', W') &\leq v \Rightarrow W' \leq p_1' \exp(v) \Rightarrow (1-\alpha) W' \leq (1-\alpha) p_1' \exp(v) \end{aligned}$$

*hence*

$$\alpha W + (1-\alpha) W' \leq \alpha p_1 \exp(v) + (1-\alpha) p_1' \exp(v)$$

$$\begin{aligned} &\Rightarrow \frac{\alpha W + (1-\alpha) W'}{\alpha p_1 + (1-\alpha) p_1'} \leq \exp(v) \\ &\Rightarrow \ln\left(\frac{\alpha W + (1-\alpha) W'}{\alpha p_1 + (1-\alpha) p_1'}\right) \leq v \end{aligned}$$

*That is,*

$$V(\alpha p_1 + (1-\alpha) p_1', \alpha p_2 + (1-\alpha) p_2', \alpha W + (1-\alpha) W') \leq v$$

*as required.*
*And in the region* $W > p_2$

$$\begin{aligned} \text{if } V(p_1, p_2, W) &= \ln p_2 - \ln p_1 + \frac{W - p_2}{p_2} \leq v \\ \text{and } V(p_1', p_2', W') &= \ln p_2' - \ln p_1' + \frac{W' - p_2'}{p_2'} \leq v \end{aligned}$$

*then*

$$\begin{aligned} \alpha[W - p_2(1+v)] &\leq \alpha p_2 \left[\ln\left(\frac{p_1}{p_2}\right)\right] \\ &\quad \textit{and} \\ (1-\alpha)[W' - p_2'(1+v)] &\leq (1-\alpha) p_2' \left[\ln\left(\frac{p_1'}{p_2'}\right)\right] \end{aligned}$$

*Dividing both inequalities by $\alpha p_2 + (1-\alpha) p_2'$ and adding, we obtain*

$$\begin{aligned}
& \frac{\alpha W + (1-\alpha) W'}{\alpha p_2 + (1-\alpha) p_2'} - (1+v) \\
\leq \; & \frac{\alpha p_2}{\alpha p_2 + (1-\alpha) p_2'} \left[ \ln \left( \frac{p_1}{p_2} \right) \right] + \frac{(1-\alpha) p_2'}{\alpha p_2 + (1-\alpha) p_2'} \left[ \ln \left( \frac{p_1'}{p_2'} \right) \right] \\
\leq \; & \ln \left( \frac{\alpha p_1 + (1-\alpha) p_1'}{\alpha p_2 + (1-\alpha) p_2'} \right).
\end{aligned}$$

*The last inequality follows from Jensen's inequality for concave functions. Thus rearranging this inequality we obtain,*

$$\ln \left( \frac{\alpha p_2 + (1-\alpha) p_2'}{\alpha p_1 + (1-\alpha) p_1'} \right) + \frac{\alpha W + (1-\alpha) W'}{\alpha p_2 + (1-\alpha) p_2'} - 1 \leq v$$

*That is,*

$$V \left( \alpha p_1 + (1-\alpha) p_1', \alpha p_2 + (1-\alpha) p_2', \alpha W + (1-\alpha) W' \right) \leq v$$

*as required.*

Consider an economy with a continuum of consumers of total measure 1. Half of the consumers are of type $I$ whose preferences can be represented by the utility function

$$u^I (x_1, x_2) = \ln x_1 + x_2.$$

The remaining half of the consumers are of type $II$ whose preferences may be represented by the utility function

$$u^{II} (x_1, x_2) = x_1 + \ln x_2.$$

Suppose every consumer has the same wealth equal to the per-capita wealth $\bar{W}$.

**(b)** Show that the *per capita* demand for goods 1 and 2, can be expressed as the following function of $p_1, p_2$ and per capita wealth $\bar{W}$ :

$$\begin{aligned}
\bar{x}_1 \left( p_1, p_2, \bar{W} \right) &= \frac{\min \left( p_2, \bar{W} \right) + \max \left( \bar{W} - p_1, 0 \right)}{2 p_1} \\
\bar{x}_2 \left( p_1, p_2, \bar{W} \right) &= \frac{\max \left( \bar{W} - p_2, 0 \right) + \min \left( p_1, \bar{W} \right)}{2 p_1}
\end{aligned}$$

Compute the per capita demand and the demands for individuals of type $I$ and type $II$ for goods 1 and 2 when $p_1 = p_2 = 1$ and per capita wealth $\bar{W} = 3/2$. (10 points).

**ANS:** *Using the answer from part (a) (and switching the roles of good 1 and 2 for type II) we have*

$$\begin{aligned}
x_1^I\left(p_1,p_2,\bar{W}\right) &= \frac{\min\left(p_2,\bar{W}\right)}{p_1}\\
x_2^I\left(p_1,p_2,\bar{W}\right) &= \frac{\max\left(\bar{W}-p_2,0\right)}{p_2}\\
x_1^{II}\left(p_1,p_2,\bar{W}\right) &= \frac{\max\left(\bar{W}-p_1,0\right)}{p_1}\\
x_2^{II}\left(p_1,p_2,\bar{W}\right) &= \frac{\min\left(p_1,\bar{W}\right)}{p_2}
\end{aligned}$$

*Adding we have*

$$\begin{aligned}
\bar{x}_1\left(p_1,p_2,\bar{W}\right) &= \frac{1}{2}x_1^I\left(p_1,p_2,\bar{W}\right)+\frac{1}{2}x_1^{II}\left(p_1,p_2,\bar{W}\right)\\
&= \frac{\min\left(p_2,\bar{W}\right)+\max\left(\bar{W}-p_1,0\right)}{2p_1}\\
\bar{x}_2\left(p_1,p_2,\bar{W}\right) &= \frac{1}{2}x_2^I\left(p_1,p_2,\bar{W}\right)+\frac{1}{2}x_2^{II}\left(p_1,p_2,\bar{W}\right)\\
&= \frac{\max\left(\bar{W}-p_2,0\right)+\min\left(p_1,\bar{W}\right)}{2p_1}
\end{aligned}$$

*Plugging in* $p_1=p_2=1$ *and* $\bar{W}=3/2$ *yields*

$$\begin{aligned}
x_1^I(1,1,3/2) &= 1\\
x_2^I(1,1,3/2) &= 1/2\\
x_1^{II}(1,1,3/2) &= 1/2\\
x_2^{II}(1,1,3/2) &= 1\\
\bar{x}_1(1,1,3/2) &= \frac{1}{2}\times 1+\frac{1}{2}\times\frac{1}{2}=3/4\\
\bar{x}_2(1,1,3/2) &= \frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times 1=3/4
\end{aligned}$$

For the remainder of the question assume all individuals have the same wealth, that is, they all have wealth equal to the per capita wealth $\bar{W}$, and further assume $\bar{W}>\max(p_1,p_2)$.

**(c)** When $\bar{W}>\max(p_1,p_2)$, show that the per capita demand you derived in part (b) is the solution to the following utility maximization problem of a representative consumer with preferences represented by the *indirect* utility function

$$\bar{V}\left(p_1,p_2,\bar{W}\right)=\frac{\bar{W}+p_1+p_2}{\sqrt{p_1p_2}}$$

$$\bar{u}(x_1,x_2)=-\left(x_1+1\right)^{-1}\left(x_2+1\right)^{-1}$$

(15 points).

**ANS:** *Recall by Roy's identity*

$$\bar{x}_\ell = -\frac{\partial \bar{V}/\partial p_\ell}{\partial \bar{V}/\partial W}$$

*Since*

$$\begin{aligned}
\frac{\partial \bar{V}}{\partial W} &= \frac{1}{\sqrt{p_1 p_2}}, \\
\frac{\partial \bar{V}}{\partial p_1} &= \frac{1}{2}\left(-\frac{\bar{W}}{p_1^{3/2}\sqrt{p_2}} + \frac{1}{\sqrt{p_1 p_2}} - \frac{\sqrt{p_2}}{p_1^{3/2}}\right), \\
\frac{\partial \bar{V}}{\partial p_2} &= \frac{1}{2}\left(-\frac{\bar{W}}{p_2^{3/2}\sqrt{p_1}} + \frac{1}{\sqrt{p_1 p_2}} - \frac{\sqrt{p_1}}{p_2^{3/2}}\right)
\end{aligned}$$

*by applying Roy's identity we obtain*

$$\begin{aligned}
\bar{x}_1\left(p_1, p_2, \bar{W}\right) &= -\frac{1}{2}\left(-\frac{\bar{W}}{p_1^{3/2}\sqrt{p_2}} + \frac{1}{\sqrt{p_1 p_2}} - \frac{\sqrt{p_2}}{p_1^{3/2}}\right) \times \sqrt{p_1 p_2} \\
&= \frac{1}{2}\left(\frac{\bar{W}}{p_1} - 1 + \frac{p_2}{p_1}\right) = \frac{\bar{W} - p_1 + p_2}{2p_1} \\
\bar{x}_2\left(p_1, p_2, \bar{W}\right) &= -\frac{1}{2}\left(-\frac{\bar{W}}{p_2^{3/2}\sqrt{p_1}} + \frac{1}{\sqrt{p_1 p_2}} - \frac{\sqrt{p_1}}{p_2^{3/2}}\right) \times \sqrt{p_1 p_2} \\
&= \frac{1}{2}\left(\frac{\bar{W}}{p_2} - 1 + \frac{p_1}{p_2}\right) = \frac{\bar{W} - p_2 + p_1}{2p_2}.
\end{aligned}$$

**(d)** Derive the substitution matrices for consumers of type $I$, for consumers of type $II$ and for the representative consumer. Denote these matrices by $S^I\left(p_1, p_2, W^I\right)$ $S^{II}\left(p_1, p_2, W^{II}\right)$ and $\bar{S}\left(p_1, p_2, \bar{W}\right)$, respectively. Evaluate these matrices for $p_1 = p_2 = 1$ and $W^I = W^{II} = \bar{W} = 3/2$. Show that

$$C(1, 1, 3/2) = \frac{1}{2}S^I(1, 1, 3/2) + \frac{1}{2}S^{II}(1, 1, 3/2) - \bar{S}(1, 1, 3/2)$$

is negative definite. Explain the welfare significance of this. (40 points).

**ANS:** *Differentiating the identity*

$$h_\ell^i(p, u) \equiv x_\ell^i\left(p, e^i(p, u)\right)$$

*wrt* $p_k$ *yields the Slutsky equation:*

$$\begin{aligned}
\frac{\partial h_\ell^i}{\partial p_k} &= \frac{\partial x_\ell^i}{\partial p_k} + \frac{\partial x_\ell^i}{\partial W}\frac{\partial e^i}{\partial p_k} \\
&= \frac{\partial x_\ell^i}{\partial p_k} + \frac{\partial x_\ell^i}{\partial W}h_\ell^i(p, u) \ \ \text{(by Shephard's lemma)} \\
&= \frac{\partial x_\ell^i}{\partial p_k} + \frac{\partial x_\ell^i}{\partial W}x_\ell^i(p, W), \text{ where } W = e^i(p, u)
\end{aligned}$$

*Derivating the uncompensated demand for consumers of type I wrt $p_k$ and utilizing the Slutsky equation:*

$$\begin{aligned}
\frac{\partial h_1^I}{\partial p_1} &= -\frac{p_2}{p_1^2} + 0 = -\frac{p_2}{p_1^2}, \\
\frac{\partial h_2^I}{\partial p_1} &= 0 + \frac{1}{p_2}\left(\frac{p_2}{p_1}\right) = \frac{1}{p_1}, \\
\frac{\partial h_1^I}{\partial p_2} &= \frac{1}{p_1} + 0 = \frac{1}{p_1}, \\
\frac{\partial h_2^I}{\partial p_2} &= -\frac{W}{p_2^2} + \frac{1}{p_2}\frac{(W - p_2)}{p_2} = -\frac{1}{p_2}
\end{aligned}$$

*Similarly, we can derive the substitution matrix for consumers of type $II$. So we have*

$$S^I(p_1, p_2, W) = \begin{bmatrix} -p_2/p_1^2 & 1/p_1 \\ 1/p_1 & -1/p_2 \end{bmatrix} \text{ and } S^{II}(p_1, p_2, W) = \begin{bmatrix} -1/p_1 & 1/p_2 \\ 1/p_2 & -p_1/p_2^2 \end{bmatrix}.$$

*Furthermore,*

$$\begin{aligned}
&\frac{1}{2}S^I(p_1, p_2, W) + \frac{1}{2}S^{II}(p_1, p_2, W) \\
&= \frac{(p_1 + p_2)}{2}\begin{bmatrix} -1/p_1^2 & 1/(p_1p_2) \\ 1/(p_1p_2) & -1/p_2^2 \end{bmatrix}
\end{aligned}$$

*And for the representative consumer:*

$$\begin{aligned}
\frac{\partial \bar{h}_1}{\partial p_1} &= -\left(\frac{W + p_2}{2p_1^2}\right) + \frac{1}{2p_1}\left(\frac{W + p_2 - p_1}{2p_1}\right) \\
&= -\left(\frac{W + p_1 + p_2}{4p_1^2}\right) \\
\frac{\partial \bar{h}_2}{\partial p_1} &= \frac{1}{2p_2} + \frac{1}{2p_2}\left(\frac{W + p_2 - p_1}{2p_1}\right) \\
&= \left(\frac{W + p_2 + p_1}{4p_1p_2}\right) \\
\frac{\partial \bar{h}_1}{\partial p_2} &= \frac{1}{2p_1} + \frac{1}{2p_1}\left(\frac{W + p_1 - p_2}{2p_2}\right) \\
&= \left(\frac{W + p_2 + p_1}{4p_1p_2}\right) \\
\frac{\partial \bar{h}_2}{\partial p_2} &= -\left(\frac{W + p_1}{2p_2^2}\right) + \frac{1}{2p_2}\left(\frac{W + p_1 - p_2}{2p_2}\right) \\
&= -\left(\frac{W + p_1 + p_2}{4p_2^2}\right)
\end{aligned}$$

*So the substitution matrix for the representative consumer is given by*

$$\bar{S}\left(p_1, p_2, \bar{W}\right) = \frac{\left(\bar{W} + p_1 + p_2\right)}{4}\begin{bmatrix} -1/p_1^2 & 1/(p_1p_2) \\ 1/(p_1p_2) & -1/p_2^2 \end{bmatrix}$$

*Hence constructing the matrix*

$$\begin{aligned} C\left(p_1,p_2,\bar{W}\right) &= \frac{1}{2}S^I\left(p_1,p_2,\bar{W}\right)+\frac{1}{2}S^{II}\left(p_1,p_2,\bar{W}\right)-\bar{S}\left(p_1,p_2,\bar{W}\right) \\ &= \frac{\left(\bar{W}-p_1-p_2\right)}{4}\begin{bmatrix} 1/p_1^2 & -1/\left(p_1p_2\right) \\ -1/\left(p_1p_2\right) & 1/p_2^2 \end{bmatrix} \end{aligned}$$

*We see for $\bar{W}=3/2$ and $p_1=p_2=1$,*

$$\begin{aligned} S^I\left(1,1,3/2\right) &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = S^{II}\left(1,1,3/2\right), \\ \bar{S}\left(1,1,3/2\right) &= \frac{7}{8}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \\ \text{\& hence } C\left(1,1,3/2\right) &= \frac{1}{8}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \text{ which is negative definite!} \end{aligned}$$

***[An alternative way (that some of you did) is to use the indirect utility functions to find the expenditure functions of each type of individual and the representative consumer and then calculate the substitution matrices directly by finding the Hessians of each.]***

*The significance of this is that although the representative consumer exists, it cannot be made normative for any social welfare function. In particular if $C\left(1,1,3/2\right)$ is not positive definite, then for the boundaries of the following two sets which both include $\bar{x}\left(1,1,3/2\right)=\left(3/4,3/4\right)$ :*

$$\begin{aligned} A &= \left\{\bar{x}=\frac{1}{2}x^I+\frac{1}{2}x^{II} : u^I\left(x^I\right)\geq u^I\left(3/4,3/4\right) \text{ and } u^{II}\left(x^{II}\right)\geq u^{II}\left(3/4,3/4\right)\right\} \\ \text{and } B &= \left\{\left(\bar{x}_1,\bar{x}_2\right) : \bar{u}\left(\bar{x}_1,\bar{x}_2\right)\geq \bar{u}\left(\bar{x}_1\left(1,1,3/2\right),\bar{x}_2\left(1,1,3/2\right)\right)\right\} \end{aligned}$$

*the curvature of $B$ is greater than $A$. This means it is possible to find a per-capita bundle $\bar{x}'$ such that*

$$\bar{u}\left(3/4,3/4\right)<\bar{u}\left(\bar{x}'\right)$$

*and yet for which there exists an allocation among the two types of consumers such that*

$$\frac{1}{2}\hat{x}^I+\frac{1}{2}\hat{x}^{II}=\bar{x}'$$

*and for which both*

$$u^I\left(\hat{x}^I\right)>u^I\left(3/4,3/4\right) \text{ and } u^{II}\left(\hat{x}^I\right)>u^{II}\left(3/4,3/4\right), \text{ hold.}$$

*That is, according to the representative consumer, the alternative per capita bundle $\bar{x}'$ is ranked inferior to the per capita bundle $\bar{x}$ and yet there exists an allocation across consumers yielding the same per capita consumption which makes both types of consumer strictly better off than they are under $\bar{x}$.*

2. **[30 Points]** An individual taxpayer has an income $y$ that he should report to the tax authority. Tax is payable at a constant proportionate rate $t$. The taxpayer reports $x$

where $0 \leq x \leq y$ and is aware that the tax authority audits some tax returns. Assume that the probability that the taxpayer's report is audited is $\pi$, that when an audit is carried out the true taxable income becomes public knowledge and that, if $x < y$, the taxpayer must pay both the underpaid tax and a surcharge of $s$ times the underpaid tax.

**(a)** If the taxpayer chooses $x < y$, show that disposable income $c$ in the two mutually exclusive events $NA$ (taxpayer is not audited) and $A$ (taxpayer is audited) is given by

$$\begin{aligned} c_{NA} &= y - tx \\ c_A &= (1 - t - st)\, y + stx \end{aligned}$$

(5 points).

***ANS***. *If the taxpayer chooses $x < y$ and he is not audited then his disposable income is $y$ less the assessed tax on the $x$ that he declares, i.e. $tx$, so*

$$c_{NA} = y - tx$$

. *If he is audited then he pays tax assessed on his actual income i.e. $ty$, plus the surcharge $s \times t\,(y - x)$, so*

$$\begin{aligned} c_A &= y - ty - st\,(y - x) \\ &= (1 - t - st)\, y + stx \end{aligned}$$

Assume that the individual is an expected utility maximizer with a preference scaling function over consumption of $u\,(.)$, where $u$ is increasing and strictly concave.

**(b)** Write down the first order necessary condition for an interior maximum. Explain why or why not this condition is sufficient for an interior maximum. (5 points).
***ANS***. *Program is*

$$\max_{\langle x \rangle} (1 - \pi)\, u\,(y - tx) + \pi u\,((1 - t - st)\, y + stx)$$

*FOC for interior solution:*

$$x : -(1 - \pi)\, tu'\,(y - tx^*) + \pi stu'\,((1 - t - st)\, y + stx^*) = 0$$

*or, equivalently,*

$$\frac{(1 - \pi)\, u'\,(y - tx^*)}{\pi u'\,((1 - t - st)\, y + stx^*)} = s$$

*That is, the marginal rate of substitution between consumption in the non-audit event to consumption in the audit event is $s$.* ***DRAW A PICTURE IN STATE-CONTINGENT CONSUMPTION SPACE TO ILLUSTRATE THIS SOLUTION.***

*This FOC is sufficient since the preference scaling function is strictly concave which in turn means that (expected utility) preferences over state-contingent consumption is convex. Further notice we can see directly that the SOC is satisfied since*

$$(1 - \pi)\, tu''\,(y - tx^*) + \pi stu''\,((1 - t - st)\, y + stx^*) < 0, \text{ since } u'' < 0$$

**(c)** Show that if $1 - \pi - \pi s > 0$ then the individual will definitely under-report income. (5 points).

***ANS.***

$$1 - \pi - \pi s > 0 \Rightarrow \frac{1-\pi}{\pi s} > 1$$

*Hence from FOC we have*

$$\frac{u'(y - tx^*)}{u'((1-t-st)\,y + stx^*)} = \frac{\pi s}{1-\pi} < 1$$

*or*

$$u'(y - tx^*) < u'((1-t-st)\,y + stx^*)$$

*which in turn implies (since $u' > 0$ and $u'' < 0$)*

$$\begin{aligned} & y - tx^* > (1-t-st)\,y + stx^* \\ \Rightarrow\ & (t + st)\,y > (t+st)\,x^* \\ \Rightarrow\ & y > x^*. \end{aligned}$$

Assume for the rest of the question that the optimal report $x^*$ satisfies $0 < x^* < y$.

**(d)** Show that if the surcharge is raised then under-reported income will decrease. (5 points).

***ANS.*** *Differentiating FOC wrt $s$ yields*

$$(1-\pi)\,u''(y - tx^*)\,\frac{dx^*}{ds} + \pi s u''((1-t-st)\,y + stx^*)\left(-y + x^* + s\frac{dx^*}{ds}\right) = 0$$

*Rearranging, yields*

$$\frac{dx^*}{ds} = \frac{(y - x^*)\,\pi s u''((1-t-st)\,y + stx^*)}{(1-\pi)\,u''(y - tx^*) + \pi s u''((1-t-st)\,y + stx^*)} > 0$$

*That is, increasing the penalty surcharge increases the reporting of income.-*

**(e)** If true income increases, will under-reported income increase or decrease? Briefly explain the reason for your answer? [Hint: What property of the preference scaling function will this 'wealth' effect depend upon?] (10 points).

***ANS.*** *Diagramatically, we can see if preferences are CARA then the wealth expansion path is LINEAR and PARALLEL to the certainty line which in this context implies that under-reported income (i.e. $y - x$) is constant. Show this formally by considering the FOC condtion from part (b) for a CARA preference scaling function $u(c) = -\exp(-ac)$. So if preferences exhibit DARA then $y - x$ will be increasing in $y$.*

*More formally, differentiating FOC wrt $y$ yields*

$$(1-\pi)\,tu''(y - tx^*)\left[1 - t\frac{dx^*}{dy}\right] = \pi stu''((1-t-st)\,y + stx^*)\left[(1-t-st) + st\frac{dx^*}{dy}\right]$$

*By dividing the LHS by* $(1-\pi)\,tu'(y-tx^*)$ *and the RHS by* $\pi stu'((1-t-st)\,y+stx^*)$ *(Q. Why can I do this?), we obtain*

$$R_a(C_{NA})\left[1-t\frac{dx^*}{dy}\right]=R_a(C_A)\left[(1-t-st)+st\frac{dx^*}{dy}\right]$$

*where* $R_a(c)=-u''(c)/u'(c)$ *is the coefficient of relative risk aversion. Or,,*

$$R_a(C_{NA})[1-t]+tR_a(C_{NA})\frac{d(y-x^*)}{dy}=R_a(C_A)[1-t]-stR_a(C_A)\frac{d(y-x^*)}{dy}$$

*Hence*

$$\frac{d(y-x^*)}{dy}=\frac{(R_a(C_A)-R_a(C_{NA}))(1-t)}{t(R_a(C_{NA})+sR_a(C_A))}$$

*The sign of* $d(y-x^*)/dy$ *depends on the sign of* $R_a(C_A)-R_a(C_{NA})$. *In particular, if* $u(.)$ *exhibits DARA then we see that* $d(y-x^*)/dy>0$, *since* $C_A<C_{NA}$, *implies for DARA preferences that* $R_a(C_A)>R_a(C_{NA})$.

3. **[15 Points]** For any homothetic production function show that the cost function must be expressible in the form

$$c(w,q)=c(w,1)\,h(q)\,,$$

where $h(.)$ is an increasing function and $c(w,1)$ is concave in $w$.

***ANS***. *By definition*

$$c(w,q)=\min_{\langle z\geq 0\rangle} w\cdot z \;\; s.t. \;\; f(z)\geq q \quad \text{Pblm 1}$$

*and*

$$c(w,1)=\min_{\langle z\geq 0\rangle} w\cdot z \;\; s.t. \;\; f(z)\geq 1 \; \text{Pblm 2}$$

*Now homotheticity of* $f$ *means for any pair of input vectors* $z$ *and* $z'$ *and any scalar* $\lambda>0$

$$f(z)\geq f(z')\Rightarrow f(\lambda z)\geq f(\lambda z')$$

*CLAIM: If* $z^*$ *is solution to* Pblm 2 *then* $h(q)\,z^*$ *is solution to* Pblm 1, *where* $h(q)$ *is solution to*

$$f(h(q)\,z^*)=q.$$

*Proof: Suppose not. That is, suppose there exists* $\hat{z}$ *such that* $f(\hat{z})\geq q$ *and* $w\cdot\hat{z}<w\cdot(h(q)\,z^*)$. *But since* $f(\hat{z})\geq f(h(q)\,z^*)$ *it follows from homotheticity of* $f(.)$ *that* $f(\hat{z}/h(q))\geq f(z^*)=1$ *and*

$$w\cdot[\hat{z}/h(q)]<w\cdot z^*=c(w,1)\,,\; \textit{a contradiction.}$$

*The claim has established that* $h(q)\,z^*$ *is a solution to* Pblm 1. *Thus we have,*

$$c(w,q)=w\cdot(h(q)\,z^*)=h(q)\,(w\cdot z^*)=h(q)\,c(w,1)\,,$$

*as required.*

*It remains to show $c(w,1)$ is concave. Let $z'$ (respectively, $z''$, $z_\alpha$) be a cost-minimizing input vector for prices $w'$ (respectively, $w''$, $\alpha w' + (1-\alpha) w''$). Hence*

$$\begin{aligned} w'.z' &\le w'.z_\alpha \Rightarrow \alpha w'.z' \le \alpha w'.z_\alpha \\ w''.z'' &\le w''.z_\alpha \Rightarrow (1-\alpha) w''.z'' \le (1-\alpha) w''.z_\alpha \end{aligned}$$

*Adding*

$$\begin{aligned} & \alpha w'.z' + (1-\alpha) w''.z'' \le \alpha w'.z_\alpha + (1-\alpha) w''.z_\alpha \\ \Rightarrow \; & \alpha c(w,1) + (1-\alpha) c(w,1) \le (\alpha w' + (1-\alpha) w'') .z_\alpha = c(\alpha w' + (1-\alpha) w'', 1), \end{aligned}$$

*as required.*

4. **[15 Points]** Consider an economy with a fixed number of firms, each characterized by its production set. Suppose the standard assumptions hold. In particular, suppose there are no production externalities. That is, the production possibilities available to one firm are unaffected by the production plan adopted by any other firm. Suppose all firms are price-takers. Let $y^0$ denote the *aggregate* supply associated with prices $p^0$ and let $y^1$ denote the *aggregate* supply associated with prices $p^1$. Assuming all firms are profit-maximizers state and prove the relationship that must hold between $(p^0, y^0)$ and $(p^1, y^1)$.

   ***ANS***. *Want to show AGGREGATE LAW OF SUPPLY holds, namely*

$$\left(p^1 - p^0\right) \cdot \left(y^1 - y^0\right) \ge 0.$$

   *Note by definition $y^0 = \sum_j y_j^0$ and $y^1 = \sum_j y_j^1$, where for each $j$, $y_j^0$ (respectively, $y_j^1$) is the supply of firm $j$ associated with prices $p^0$, (respectively, $p^1$). Since each firm is a price-taking profit maximizer it follows that*

$$\begin{aligned} p^0 \cdot y_j^0 &\ge p_j^0 \cdot y_j \text{ for all } y_j \in Y_j, \\ \text{and } p^1 \cdot y_j^1 &\ge p_j^1 \cdot y_j \text{ for all } y_j \in Y_j \end{aligned}$$

   *So in particular, we have,*

$$\begin{aligned} p^0 \cdot y_j^0 &\ge p_j^0 \cdot y_j^1 \Rightarrow -p^0 \cdot \left(y_j^1 - y_j^0\right) \ge 0 \\ \text{and } p^1 \cdot y_j^1 &\ge p_j^1 \cdot y_j^0 \Rightarrow p^1 \cdot \left(y_j^1 - y_j^0\right) \ge 0 \end{aligned}$$

   *Adding gives us the law of supply for each firm $j$,*

$$\left(p^1 - p^0\right) \cdot \left(y_j^1 - y_j^0\right) \ge 0$$

   *Summing over $j$,*

$$\begin{aligned} & \sum\nolimits_j \left(p^1 - p^0\right) \cdot \left(y_j^1 - y_j^0\right) \ge 0 \\ \Rightarrow \; & \left(p^1 - p^0\right) \cdot \left[\sum\nolimits_j \left(y_j^1 - y_j^0\right)\right] \ge 0 \\ \Rightarrow \; & \left(p^1 - p^0\right) \cdot \left(\sum\nolimits_j y_j^1 - \sum\nolimits_j y_j^0\right) \ge 0 \\ \Rightarrow \; & \left(p^1 - p^0\right) \cdot \left(y^1 - y^0\right) \ge 0. \end{aligned}$$

5. **[30 Points]** A government owned enterprise (GOE) generates electricity for the town of Wagga with a constant returns to scale technology with constant marginal cost of electricity generation equal to $c$. There are two-types of households who demand electricity in Wagga. The fraction $\lambda$ are $H$-types, while the remaining fraction $(1-\lambda)$ are $L$ -types. If an $H$-type (respectively, $L$-type) household consumed $q$ units of electricity and paid $T$ in total then the consumer surplus enjoyed by that household is given by

$$\begin{aligned} CS_H(q,T) &= u_H(q) - T \\ (\text{respectively, } CS_L(q,T) &= u_L(q) - T), \end{aligned}$$

where $u_H$ and $u_L$ are both increasing, twice continuously differentiable and strictly concave functions with $u_H(0) = u_L(0) = 0$, $u_H'(0) = \alpha$, $u_H'(\bar{q}) = 0$, and $u_H'(q) > u_L'(q)$ for all $q \in [0, \bar{q})$.

For this question assume that the GOE cannot distinguish $H$ type households from $L$ type households.

**(a)** Design a pricing scheme that maximizes the sum of consumer and producer surplus. (7 points).

**ANS:** *Remember the first fundamental welfare theorem for partial equilibrium, social surplus is maximized where price is equal to marginal cost. More formally, as a function of $q_H$, the quantity consumed by $H$-type households and $q_L$ the quantity consumed by L-type households, the sum of consumer and producer surplus may be written as*

$$\lambda(u_H(q_H) - cq_H) + (1-\lambda)(u_L(q_L) - cq_L).$$

*So the first-best is the solution to the following program:*

$$\max_{\langle q_L, q_H\rangle} \lambda(u_H(q_H) - cq_H) + (1-\lambda)(u_L(q_L) - cq_L)$$

*First order necessary (and sufficient) conditions*

$$\begin{aligned} q_H &: u_H'(q_H^*) = c \\ q_L &: u_L'(q_L^*) = c \end{aligned}$$

*But this can be implemented by setting a price $p = c$. Notice the utility maximization problem for $H$-type households is then:*

$$\max_{\langle q_H\rangle} u_H(q_H) - cq_H$$

*and so they will choose the quantity $q_H^*$ since it satisfies the FOC*

$$u_H'(q_H^*) = p\ (= c)$$

*Similarly, L-type Notice that profits are zero, at this allocation that maximizes the sum of consumer surplus and producer surplus.*

Now suppose the government faces an excess burden in raising revenue from the imposition of distorting taxes in other markets.

**(b)** If the marginal excess burden of raising a dollar of revenue is $\gamma > 0$, explain why the *social opportunity cost* to the government of forgoing a dollar in profit from electricity generation is $1+\gamma$. (3 points).

*Every dollar of profit raised from electricity generation allows the government to reduce distortionary taxes by an amount so that revenue from distortionary taxes is reduced by a dollar. The "equivalent" variation of this reduction in distortionary taxes is $1+\gamma$ to consumers.*

**(c)** Design a non-linear pricing scheme that maximizes

$$\lambda CS_H + (1-\lambda)\, CS_L + (1+\gamma)\, \pi$$

where recall $CS_H$ (respectively, $CS_L$) is the consumer surplus enjoyed by an $H$-type (respectively, $L$-type) household and $\pi$ is the *per-household* profit earned by the GOE.

[Hint: Find the 'optimal' two-element "menu" $\left(\hat{q}_H, \hat{T}_H\right)$ and $\left(\hat{q}_L, \hat{T}_L\right)$, where the first package is designed for $H$ type households and the second package is designed for $L$ type households. You should be able to show (diagrammatically) that $u_H'(\hat{q}_H) = c$ and $u_L'(\hat{q}_L) > c$. To get the first order condition that $\hat{q}_L$ must satisfy, think about the tradeoffs in lost profit from supplying an 'inefficient' amount of electricity to $L$-type households and the reduction in the 'information rent' that has to be 'paid' to the $H$-type households to get them to select the package $\left(\hat{q}_H, \hat{T}_H\right)$.] (20 points).

*This is like the optimal screening scheme for a profit-maximizing monopolist, except that unlike the monopolist, the GOE also values consumer surplus (although with less weight than is placed on profit). For a menu $(q_H, T_H)$ and $(q_L, T_L)$, if the $H-$type households choose the package $(q_H, T_H)$ and the $L$-type households selected the package $(q_L, T_L)$ then the social surplus would be*

$$\begin{aligned} & \lambda\left[u_H(q_H) - T_H\right] + (1-\lambda)\left[u_L(q_L) - T_L\right] + (1+\gamma)\left[\lambda(T_H - cq_H) + (1-\lambda)(T_L - cq_L)\right] \\ = \; & \lambda\left(u_H(q_H) - cq_H\right) + (1-\lambda)\left(u_L(q_L) - cq_L\right) + \gamma\left[\lambda(T_H - cq_H) + (1-\lambda)(T_L - cq_L)\right] \end{aligned}$$

*Formally, the program is*

$$\max_{\langle (q_H,T_H),(q_L,T_L)\rangle} \lambda\left(u_H(q_H) - cq_H\right) + (1-\lambda)\left(u_L(q_L) - cq_L\right) + \gamma\left[\lambda(T_H - cq_H) + (1-\lambda)(T_L - cq_L)\right]$$

*Subject to:*

*Individual rationality constraints,*

$$\begin{aligned} u_H(q_H) - T_H & \geq 0 \\ u_L(q_L) - T_L & \geq 0 \end{aligned}$$

*and self-selection constraints*

$$\begin{aligned} u_H(q_H) - T_H & \geq u_H(q_L) - T_L \\ u_L(q_L) - T_L & \geq u_L(q_H) - T_H \end{aligned}$$

*Recall, the IRC for $H$-type households and the SSC for $L$-type households will not be binding. If you draw the diagram you will see that you want the $H$-type households to consume an efficient amount of the good (i.e. $\hat{q}_H = q_H^*$, so that $u'_H(\hat{q}_H) = c$) but you will want to distort the low types to reduce the information rent you pay to the $H$-type households. You want to reduce the information rent paid to $H$-type households as a dollar of profit is valued more by the government than is a dollar of consumer surplus to households.*

*Draw the diagram and you will see that the optimal menu takes the form:*

$$
\begin{aligned}
&\hat{q}_H \text{ is such that } u'_H(\hat{q}_H) = c, \\
&\hat{T}_H = u_H(q_H) - [u_H(\hat{q}_L) - u_L(\hat{q}_L)] \\
&\hat{T}_L = u_L(\hat{q}_L),
\end{aligned}
$$

*where $\hat{q}_L$ satisfies*

$$\gamma\lambda [u'_H(\hat{q}_L) - u'_L(\hat{q}_L)] = (1+\gamma)(1-\lambda)[u'_L(\hat{q}_L) - c]$$

*To interpret this condition, notice that $u'_L(\hat{q}_L) - c$ is the rate at which profit earned from the L-type households is falling as the GOE reduces the quantity offered to the L-type households. This is weighted by $(1+\gamma)$ since a dollar of profit has a social value greater than 1 because $\gamma$ is the marginal excess burden of raising a dollar through (distortionary) taxation and also by $(1-\lambda)$ the fraction of the households that are L-types. This has to be balanced against the marginal benefit of reducing the quantity offered to the L-type households, which is $u'_H(\hat{q}_L) - u'_L(\hat{q}_L)$ (the rate at which the information rent that is paid to the $H$-type households falls) times $\gamma$ (the difference in social value between a dollar going to the $H$-type households and going to the government as profit) times $\lambda$ the fraction of the households that are $H$-types.*

*Intuitively, we return to marginal cost pricing if $\gamma = 0$, and as $\gamma \to \infty$, the optimal menu converges to the optimal screening scheme for the profit-maximizing monopolist.*