Rice University

Answer Key to Mid-Semester Examination Fall 2006

ECON 501: Advanced Microeconomic Theory

Part A

1. Consider the following expenditure function.

$$e(p_1, p_2, p_3, u) = (p_1 + 2p_2)\, u + p_3$$

State the properties an indirect utility must satisfy and show that the indirect utility function that can be derived from the above expenditure function satisfies all of them.

You all knew the properties and you knew how to derive the indirect utility function using the identity:

$$e(p_1, p_2, V(p_1, p_2, p_3, w)) \equiv w$$

which yielded

$$V(p_1, p_2, p_3, w) = \frac{w - p_3}{p_1 + 2p_2}$$

You also all had no difficulty in showing this indirect utility function satisfied all the properties required of an indirect utility function except the property of quasiconvexity with respect to prices and wealth. But one of you provided a very neat argument. It begins by noting, quasiconvexity means that for each u *in the range of* $V(p_1, p_2, p_3, w)$*, the set*

$$D = \left\{(p_1, p_2, p_3, w) \in \mathbb{R}^3_{++} : V(p_1, p_2, p_3, w) \leq u\right\}$$

is a convex set. So we need to show if $(p_1', p_2', p_3', w') \in D$ *and* $(p_1'', p_2'', p_3'', w'') \in D$ *then*

$$(\alpha p + [1-\alpha]\, p, \alpha w + [1-\alpha]\, w'') \in D, \textit{ for all } \alpha \in [0, 1].$$

But if $(p_1', p_2', p_3', w') \in D$ *and* $(p_1'', p_2'', p_3'', w'') \in D$*, then*

$$\begin{aligned}
& \frac{w' - p_3'}{p_1' + 2p_2'} \leq u \textit{ and } \frac{w'' - p_3''}{p_1'' + 2p_2''} \leq u \\
\Leftrightarrow\ & \alpha(w' - p_3') \leq \alpha(p_1' + 2p_2')\, u \textit{ and } (1-a)(w'' - p_3'') \leq (1-\alpha)(p_1'' + 2p_2'')\, u \\
\Leftrightarrow\ & \alpha(w' - p_3') + (1-a)(w'' - p_3'') \leq [\alpha(p_1' + 2p_2') + (1-\alpha)(p_1'' + 2p_2'')]\, u \\
\Leftrightarrow\ & \frac{\alpha w' + [1-\alpha]\, w'' - (\alpha p_3' + [1-\alpha]\, p_3'')}{[\alpha(p_1' + 2p_2') + (1-\alpha)(p_1'' + 2p_2'')]} \leq u \\
\Rightarrow\ & (\alpha p + [1-\alpha]\, p, \alpha w + [1-\alpha]\, w'') \in D \textit{ as required.}
\end{aligned}$$

2. For the underlying preferences implicitly defined in Q1, derive the uncompensated demand and the compensated demand and verify that the Slutsky equation holds.

You knew to use Roy's identity to derive the uncompensated demand and Shepherd's lemmas to derive the uncompensated.

(a) *Uncompensated demands*

$$\frac{\partial V}{\partial w}=\frac{1}{p_1+2p_2},\ \frac{\partial V}{\partial p_1}=\frac{-(w-p_3)}{(p_1+2p_2)^2},\ \frac{\partial V}{\partial p_1}=\frac{-2(w-p_3)}{(p_1+2p_2)^2},\ \frac{\partial V}{\partial p_1}=\frac{-1}{p_1+2p_2}$$

So

$$\begin{aligned}
x_1(p,w) &= -\frac{\partial V/\partial p_1}{\partial V/\partial w}=\frac{(w-p_3)}{(p_1+2p_2)}\\
x_2(p,w) &= -\frac{\partial V/\partial p_2}{\partial V/\partial w}=\frac{2(w-p_3)}{(p_1+2p_2)}\\
x_3(p,w) &= -\frac{\partial V/\partial p_3}{\partial V/\partial w}=1
\end{aligned}$$

(b) *Compensated demands*

$$\begin{aligned}
h_1(p,u) &= \frac{\partial e}{\partial p_1}=u\\
h_2(p,u) &= \frac{\partial e}{\partial p_2}=2u\\
h_3(p,u) &= \frac{\partial e}{\partial p_3}=1
\end{aligned}$$

(c) *Slutsky's equation: for all* $\ell, k = 1, 2, 3$

$$\frac{\partial h_\ell}{\partial p_k}=\frac{\partial x_\ell}{\partial p_k}+\frac{\partial x_\ell}{\partial w}x_k$$

Notice that for all $\ell, k = 1, 2, 3$, $\partial h_\ell/\partial p_k = 0$, *i.e. the substitution matrix* $S = \mathbf{0}$. *So we need to verify*

$$\frac{\partial x_\ell}{\partial p_k}+\frac{\partial x_\ell}{\partial w}x_k=0,\ \textit{for all}\ \ell,k=1,2,3.$$

For example, for $\ell = 1$ *and* $k = 2$, *we have*

$$\frac{\partial x_1}{\partial p_2}+\frac{\partial x_1}{\partial w}x_2=-\frac{2(w-p_3)}{(p_1+2p_2)^2}+\frac{1}{(p_1+2p_2)}\times\frac{2(w-p_3)}{(p_1+2p_2)}=0,$$

as required. Other eight expressions follow in a similar way.

3. State the over-compensated law of demand for a world with $L > 1$ commodities (Recall, if x^0 is chosen when consumer faces prices p^0 and has wealth w^0, and x^1 is chosen when facing prices p^1 and wealth w^1, 'over-compensation' refers to the fact that $w^1 = p^1.x^0$). Draw a diagram illustrating its implication in a two-commodity world. Prove the over-compensated law of demand holds in a world with $L > 1$ commodities for a consumer whose choice behavior respects the Weak Axiom of Revealed Preference.

   I thought this question was a 'gimme'. Notice this is something we did in the very first lecture of the course and was revisited when we studied the theory of revealed preference.

The over-compensated law of demand states that if x^0 is chosen by a consumer when facing prices p^0 and wealth w^0, and x^1 is chosen by the same consumer when facing prices p^1 and wealth $w^1 = p^1.x^0$ (i.e. a wealth just sufficient to allow the consumer to purchase the bundle x^0) then

$$\left(p^1 - p^0\right).\left(x^1 - x^0\right) \leq 0$$

The weak axiom of revealed preference states: if x' is chosen by a consumer when facing prices p' and wealth w', and x'' is chosen by the same consumer when facing prices p'' and wealth w'', then

$$\begin{aligned} p''.x' &\leq w'' \Rightarrow p'.x'' \geq w' \\ p''.x' &< w'' \Rightarrow p'.x'' > w' \end{aligned}$$

So for the situation above, since $w^1 = p^1.x^0$, the weak axiom implies $p^0.x^1 \geq w^0$. But

$$\begin{aligned} w^1 &= p^1.x^1 = p^1.x^0 \Rightarrow p^1.\left(x^1 - x^0\right) = 0, \text{ and} \\ p^0.x^1 &\geq w^0 = p^0.x^0 \Rightarrow -p^0.\left(x^1 - x^0\right) \leq 0, \end{aligned}$$

Adding, yields

$$\left(p^1 - p^0\right).\left(x^1 - x^0\right) \leq 0, \text{ as required.}$$

Part B

4. Bernard's preferences are defined over three commodites wine, bread and leisure.

   Let x_1 denote his consumption of wine, x_2 his consumption of bread, x_3 his consumption of leisure. Suppose his consumption set is the vector space $\mathbb{R}_+ \times \mathbb{R}_+ \times [0, H]$, where $0 < H$, and suppose his preferences can be represented by the utility function

   $$U(x_1, x_2, x_3) = \frac{x_1^{\alpha} x_2^{(1-\alpha)}}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}} + \ln x_3$$

   Bernard is endowed with H hours of leisure and M dollars of non-wage income. He can work as many hours L as he wants at W dollars per hour and he faces linear prices p_1 and p_2 for wine and bread, respectively.

   (a) [**10 points**] If Bernard were to spend E dollars on wine and bread, show that the optimal way for him to allocate his expenditure of E dollars between wine and bread is independent of his consumption of leisure. Show that the maximum 'utility' he can generate by the expenditure of E dollars on wine and bread is given by the function

   $$v(p_1, p_2, E) = \frac{E}{p_1^{\alpha} p_2^{1-\alpha}}$$

Problem can be seen to be a standard UMP with Cobb-Douglas preferences:

$$\max_{\langle x_1, x_2 \rangle} u(x_1, x_2) = \frac{x_1^{\alpha} x_2^{(1-\alpha)}}{\alpha^{\alpha} (1-\alpha)^{(1-\alpha)}} \text{ s.t. } p_1 x_1 + p_2 x_2 \leq E$$

Form Lagrangean and obtain FONCs.

$$\begin{aligned}
\mathcal{L} &= u(x_1, x_2) - \lambda (p_1 x_1 + p_2 x_2 - E) \\
x_1 &: \frac{\partial \mathcal{L}}{\partial x_1} = \alpha \frac{u(x_1^*, x_2^*)}{x_1} - \lambda^* p_1 \leq 0, \text{ with equality if } x_1^* > 0 \\
x_2 &: \frac{\partial \mathcal{L}}{\partial x_2} = (1-\alpha) \frac{u(x_1^*, x_2^*)}{x_2^*} - \lambda^* p_2 \leq 0, \text{ with equality if } x_2^* > 0 \\
\lambda &: p_1 x_1^* + p_2 x_2^* - E \leq 0, \text{ with equality if } \lambda^* > 0.
\end{aligned}$$

For interior solution, first two FONC yield

$$\left(\frac{\alpha}{1-\alpha}\right) p_2 x_2^* = p_1 x_1^*$$

Substituting into the expenditure constraint:

$$\left(1 + \frac{\alpha}{1-\alpha}\right) p_2 x_2^* = E \Rightarrow x_2^* = \frac{(1-\alpha) E}{p_2}$$

Hence

$$x_1^* = \frac{\alpha E}{p_1}.$$

Plugging into the utility function $u(x_1, x_2)$ yields

$$\begin{aligned}
v(p_1, p_2, E) &= \frac{\left(\frac{\alpha E}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)E}{p_2}\right)^{(1-\alpha)}}{\alpha^{\alpha} (1-\alpha)^{(1-\alpha)}} \\
&= \frac{E}{p_1^{\alpha} p_2^{1-\alpha}}, \text{ \textit{as required.}}
\end{aligned}$$

(b) **[10 points]** Set $X := v(p_1, p_2, E)$ to be the (real) consumption of Bernard and set $P := p_1^{\alpha} p_2^{1-\alpha}$, to be the 'price' of (real) consumption. Explain why (X^*, L^*), Bernard's optimal amount of consumption and his optimal supply of labor, is the solution to the following constrained maximization problem:

$$\max_{\langle X, L \rangle} X + \ln(H - L)$$

$$\text{s.t. } PX \leq M + WL, \ X \geq 0, \ L \in [0, H]$$

Notice that $H - L = x_3$ and using our result from part (a) we have

$$X = v(p_1, p_2, E) = \max_{\langle x_1, x_2 \rangle} u(x_1, x_2) \text{ s.t. } p_1 x_1 + p_2 x_2 \leq E$$

Furthermore, $U(x_1, x_2, x_3)$ *is (additively) separable between* (x_1, x_2)*, and* x_3. That is,

$$U(x_1, x_2, x_3) = u(x_1, x_2) + \ln x_3$$

So as we showed in class for weakly separable preferences,

$$\max_{\langle x_1, x_2, x_3 \rangle} U(x_1, x_2, x_3) \text{ s.t. } p_1 x_1 + p_2 x_2 + W x_3 \leq M + WH$$

$$[\text{or equivalently s.t., } p_1 x_1 + p_2 x_2 \leq M + W(H - x_3)]$$

is equivalent to

$$\max_{\langle X, L \rangle} v(p_1, p_2, E) + \ln(H - L - h) \text{ s.t. } E = PX \leq M + WL.$$

And we can obtain $x_3^* = H - L^*$*, and using Roy's identity*

$$\begin{aligned} x_1^* &= -\frac{\partial v(p_1, p_2, E^*)/\partial p_1}{\partial v(p_1, p_2, E)/\partial w} = \frac{\alpha E^*}{p_1} = \frac{\alpha P X^*}{p_1} \\ \text{and } x_2^* &= -\frac{\partial v(p_1, p_2, E^*)/\partial p_2}{\partial v(p_1, p_2, E)/\partial w} = \frac{(1-\alpha) E^*}{p_2} = \frac{(1-\alpha) P X^*}{p_2} \end{aligned}$$

(c) [**10 points**] Assuming an interior solution to the constrained maximization problem in part (b), derive as functions of the prices (P, W) and Bernard's non-wage income M, *(i)* X, his consumption, and *(ii)* L, his supply of labor.

Form Lagrangean and obtain FONCs.

$$\begin{aligned} \mathcal{L} &= X + \ln(H - L) - \Lambda(PX - M - WL) \\ X &: \frac{\partial \mathcal{L}}{\partial E} = 1 - \Lambda^* P = 0, \text{ (since we are assuming } X^* > 0\text{)}. \\ L &: \frac{\partial \mathcal{L}}{\partial L} = -\frac{1}{(H - L)} + \Lambda^* W = 0, \text{ (since we are assuming } L^* > 0\text{)}. \\ \Lambda &: PX^* - M - WL^* = 0, \text{ (since we are assuming } \Lambda^* > 0\text{)}. \end{aligned}$$

From the first two FONCs we obtain

$$L(P, W, M) = H - \frac{P}{W}$$

and plugging that into the third FONC (i.e. the 'expenditure' constraint) yields

$$X(P, W, M) = \frac{M + WH}{P} - 1$$

(d) [**5 points**] Define the function $m(P, W, U)$ to be the minimum non-wage income Bernard requires to achieve the utility U when facing a 'price of consumption' P and wage rate W. That is,

$$m(P, W, U) = \min_{\langle X, L \rangle} PX - WL \text{ s.t. } X + \ln(H - L) \geq U$$

Show that

$$m(P,W,U) = (U + 1 - \ln P + \ln W)P - WH$$

[Hint: Do not solve the constrained minimization problem directly. Think what $V(P,W,m(P,W,U))$ must be identically equal to and use that identity along with your answer to part (c).]

Using the answer from part (c) we have

$$\begin{aligned} V(P,W,M) &= X(P,W,M) + \ln(H - L(P,W,M)) \\ &= \frac{M+WH}{P} - 1 + \ln P - \ln W \end{aligned}$$

This is the maximum utility the individual can achieve when he has non-wage income of M, and faces a price of consumption P and wage rate W. Now by definition

$$V(P,W,m(P,W,U)) = U.$$

That is, if Bernard has non-wage income of $m(P,W,U)$ and faces a price of consumption P and a wage rate W, then the maximum utility he can achieve must be U. So

$$\frac{m(P,W,U)+WH}{P} - 1 + \ln P - \ln W = U$$

Hence

$$m(P,W,U) = (U + 1 - \ln P + \ln W)P - WH$$

(e) [**5 points**] Differentiate $m(P,W,U)$ with respect to P and W, respectively. To what do $\partial m(P,W,U)/\partial P$ and $\partial m(P,W,U)/\partial W$ correspond?

By the envelope theorem

$$\begin{aligned} \frac{\partial m(P,W,U)}{\partial P} &= X(P,W,U) = U - \ln P + \ln W \\ \frac{\partial m(P,W,U)}{\partial P} &= -L(P,W,U) = \frac{P}{W} - H = -L(P,W,M) \end{aligned}$$

Notice that the compensated and uncompensated labor supplies coincide for Bernard (since the non-wage income effects are zero).

(f) [**20 points**] Suppose in the initial situation $M^0 = 0$, $P = P^0$ and $W = W^0$ and there are no taxes on consumption or labor. Now suppose in the new situation the government introduces a tax on labor at the uniform rate $\tau \in (0,1)$, so the net wage Bernard receives for an hour of labor becomes $W^1 = (1-\tau)W^0$. What happens to the quantity of labor he supplies to the market and the amount he consumes? What is the deadweight loss associated with this tax on labor?

The tax leads him to supply less labor and to consume less (real) consumption. The fall in the amount of labor he supplies and the fall in his (real) consumption *are given by*

$$\begin{aligned} L\left(P^0, W^0, 0\right) - L\left(P^0, (1-\tau) W^0, 0\right) &= \frac{P^0}{W^0}\left[\frac{1}{1-\tau} - 1\right] \\ &= \frac{P^0}{W^0} \times \frac{\tau}{(1-\tau)} \\ X\left(P^0, W^0, 0\right) - X\left(P^0, (1-\tau) W^0, 0\right) &= \frac{\tau W^0 H}{P^0} \end{aligned}$$

Following lectures, a measure for the deadweight loss is given by

$$DWL = -EV - T.$$

This situation is slightly different from the one we considered in class, where the individual was choosing how to spend a given amount of wealth among the various different commodities at given fixed prices. With the standard UMP problem it made sense to evaluate the EV by

$$EV = e\left(p^0, u^1\right) - e\left(p^0, u^0\right)$$

But our consumer's utility maximization problem was to select the optimal amount of consumption X to demand and ***labor*** *L to* ***supply*** *subject to 'budget constraint'*

$$PX = M + WL$$

In this setting it makes sense to use a money-metric indirect utility function based on the function $m(P, W, U)$, defined and derived in part (c). We take the equivalent variation of the tax to be the change in non-wage income that induces the same change in utility as that induced by the tax. That is,

$$\begin{aligned} EV &= m\left(P^0, W^0, U^1\right) - m\left(P^0, W^0, U^0\right) \\ &= m\left(P^0, W^0, U^1\right) \\ &= \left(U^1 + 1 - \ln P^0 + \ln W^0\right) P^0 - W^0 H \end{aligned}$$

We have

$$\begin{aligned} U^1 &= X\left(P^0, (1-\tau) W^0, 0\right) + \ln\left(H - L\left(P^0, (1-\tau) W^0, 0\right)\right) \\ &= \frac{(1-\tau) W^0 H}{P^0} - 1 + \ln P^0 - \ln W^0 - P^0 \ln(1-\tau) \end{aligned}$$

So

$$EV = -\tau W^0 H + P^0 \ln(1-\tau)$$

and the tax revenue raised is:

$$TAX = \tau W^0 \times L\left(P^0, (1-\tau) W^0, 0\right) = \tau\left(W^0 H - \frac{P^0}{(1-\tau)}\right)$$

Thus

$$\begin{aligned}
DWL &= -EV - TAX \\
&= \tau W^0 H - P^0 \ln(1-\tau) - \tau \left(W^0 H - \frac{P^0}{(1-\tau)} \right) \\
&= P^0 \left[\frac{\tau}{1-\tau} + \ln\left(\frac{1}{1-\tau} \right) \right] \\
&\approx 2\tau P^0 \text{ (for `small } \tau\text{')}.
\end{aligned}$$