FINAL 2005

Q1

(a) $$\min_{\langle x_1, x_2 \rangle} p_1 x_1 + p_2 x_2 \quad \text{s.t.} \quad x_1^{\alpha} x_2^{1-\alpha} \geq u$$

$$x_1 \geq 0, \; x_2 \geq 0$$

Form Lagrangean $\qquad u - x_1^{\alpha} x_2^{1-\alpha}$

$$\mathcal{L} = -p_1 x_1 - p_2 x_2 - \gamma\left(\cancel{x_1^{\alpha} x_2^{1-\alpha}} - u\right)$$

FOC

$$-p_1 + \gamma \frac{u(x_1, x_2)}{\alpha x_1} \leq 0 \quad (= 0 \text{ if } x_1 > 0)$$

$$-p_2 + \gamma \frac{u(x_1, x_2)}{(1-\alpha) x_2} \leq 0 \quad (= 0 \text{ if } x_2 > 0)$$

$$u - u(x_1, x_2) \leq 0 \quad (= 0 \text{ if } \gamma > 0)$$

$$\rightarrow \alpha x_1 p_1 = \gamma u(x_1, x_2) = (1-\alpha) x_2 p_2$$

So $$x_1 = \frac{(1-\alpha) x_2 p_2}{\alpha \; p_1}$$

Hence $$\left[\frac{1-\alpha}{\alpha}\right]^{\alpha} x_2^{\alpha} \left(\frac{p_2}{p_1}\right)^{\alpha} x_2^{1-\alpha} = u$$

$$\rightarrow \qquad x_2 = \left(\frac{\alpha}{1-\alpha}\right)^{\alpha} \left(\frac{p_1}{p_2}\right)^{\alpha} u$$

and $$x_1 = \left(\frac{1-\alpha}{\alpha}\right)^{(1-\alpha)} \left(\frac{p_2}{p_1}\right)^{(1-\alpha)} u$$

<u>Check</u> $$h_1(p_1,p_2,u) = \left[\frac{1-\alpha}{\alpha}\right]^{(1-\alpha)} \left(\frac{p_2}{p_1}\right)^{(1-\alpha)} u$$

$$h_2(p_1,p_2,u) = \left[\frac{\alpha}{1-\alpha}\right]^{\alpha} \left[\frac{p_1}{p_2}\right]^{\alpha} u$$

So $\left(h_1(p_1,p_2,u)\right]^{\alpha} \left[h_2(p_1,p_2,u)\right]^{1-\alpha}$

$$= \left(\left[\frac{1-\alpha}{\alpha}\right]^{\alpha(1-\alpha)} \cancel{x} \left(\frac{p_2}{p_1}\right)^{\alpha(1-\alpha)} u^{\alpha}\right] \cdot \left[\left(\frac{\alpha}{1-\alpha}\right)^{\alpha(1-\alpha)} \left(\frac{p_1}{p_2}\right)^{\alpha(1-\alpha)}\right] u^{(1-\alpha)}$$

$= u.$ ✓

$$e(p_1,p_2,u) = p_1 h_1(p_1,p_2,u) + p_2 h_2(p_1,p_2,u)$$

$$= \left[\left(\frac{1-\alpha}{\alpha}\right)^{(1-\alpha)} \left[p_2^{(1-\alpha)} p_1^{\alpha}\right] + \left(\frac{\alpha}{1-\alpha}\right)^{\alpha} \left[p_1^{\alpha} p_2^{(1-\alpha)}\right]\right] u$$

$$= u\, p_1^{\alpha} p_2^{(1-\alpha)} \left[\left(\frac{1-\alpha}{\alpha}\right)^{(1-\alpha)} + \left(\frac{\alpha}{1-\alpha}\right)^{\alpha}\right]$$

So $e(\lambda p_1, \lambda p_2, u) = \lambda e(p_1,p_2,u)$ as required.

~~Non~~- Increasing in $p_1, p_2,$ & u.

Concave in p_1, p_2 since $p_1^{\alpha} p_2^{1-\alpha}$ and $\cancel{p_2}^{1-\alpha}$

is a concave fn

(b) $$\max_{(x_1,x_2,x_3,x_4)} \quad x_1^{1/2} x_2^{1/2} + x_3^{1/2} x_4^{1/2}$$

$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4 \leq W$$

Note $e(p_1, p_2, u) = 2u\, p_1^{1/2} p_2^{1/2}$

and $e(p_3, p_4, u) = 2u\, p_3^{1/2} p_4^{1/2}$

So $$V(p_1, p_2, p_3, p_4, W) = \begin{cases} \dfrac{W}{2(p_1^{1/2} p_2^{1/2})} & \text{if } p_1 p_2 < p_3 p_4 \\[2ex] \dfrac{W}{2(p_3^{1/2} p_4^{1/2})} & \text{if } p_1 p_2 \geq p_3 p_4 \end{cases}$$

[* We saw in class what the demand correspondence looked like]

(d) $P^0 = (1,1,1,9)$ and $W^0 = 30$

Then $x^0 = (15, 15, 0, 0)$ and $u^0 = 15$

$P^1 = (2,2,1,9)$ and $W^1 = 30$

$x^1 = (15/2, 15/2, 0, 0)$ and $u^1 = \cancel{15}\ 15/2$

$$EV = e(1,1,u^1) - e(1,1,u^0) \cancel{= 2 \times 15 - 2 \times 15/2 = 15}$$

$\cancel{EV} = 2 \times {}^{15}/_{2} - 2 \times 15 = -15.$

$T = 1 \times x_1' + 1 \times x_2' = 15$

So $DWL = 15 - 15 = 0.$

If raised to 150% $P'' = (2.5, 2.5, 1, 9)$

$P_1'' P_2'' \leq P_3'' P_4''$ so again $DWL = 0$

If raised to 300% $P''' = (4, 4, 1, 9)$

Now $P_1''' P_2''' = 16 > 9 = P_3''' P_4'''$

So. $x''' = (0, 0, 15, {}^{15}/_{9})$ and $U''' = \sqrt{15}\sqrt{\tfrac{15}{9}}$

$= \cancel{\;} 5$

$EV = e(1,1,5) - e(1,1,15)$

$= 10 - 30 = -20$

$T = 1 \times x_1''' + 1 \times x_2''' = 0$

So $DWL = -(-20) - 0 = \underline{\underline{20}}$