Some Basic Probability Concepts

2.1 Experiments, Outcomes and Random Variables

- A random variable is a variable whose value is unknown until it is observed. The *value* of a random variable results from an experiment; it is not perfectly predictable.
- A *discrete* random variable can take only a finite number of values, that can be counted by using the positive integers.
- A *continuous* random variable can take *any* real value (not just whole numbers) in an interval on the real number line.

2.2 The Probability Distribution of a Random Variable

- When the values of a discrete random variable are listed with their chances of occurring, the resulting table of outcomes is called a *probability function* or a *probability density function*.
- For a discrete random variable X the value of the probability density function $f(x)$ *is* the probability that the random variable X takes the value x, $f(x)=P(X=x)$.
- Therefore, $0 \le f(x) \le 1$ and, if X takes n values x_1, .., x_n, then $f(x_1)+f(x_2)+\cdots+f(x_n)=1$.
- For the continuous random variable Y the probability density function $f(y)$ can be represented by an *equation*, which can be described graphically by a curve. For continuous random variables the *area* under the probability density function corresponds to probability.

2.3 Expected Values Involving a Single Random Variable

2.3.1 The Rules of Summation

1. If X takes n values x_1, ..., x_n then their sum is

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

2. If a is a constant, then

$$\sum_{i=1}^{n} a = na$$

3. If a is a constant then

$$\sum_{i=1}^{n} ax_i = a\sum_{i=1}^{n} x_i$$

4. If *X* and *Y* are two variables, then

$$\sum_{i=1}^{n}(x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$$

5. If *X* and *Y* are two variables, then

$$\sum_{i=1}^{n}(ax_i + by_i) = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i$$

6. The arithmetic mean (average) of *n* values of *X* is

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Also,

$$\sum_{i=1}^{n}(x_i - \bar{x}) = 0$$

7. We often use an abbreviated form of the summation notation. For example, if *f(x)* is a function of the values of *X*,

$$\sum_{i=1}^{n} f(x_i) = f(x_1) + f(x_2) + \cdots + f(x_n)$$

$$= \sum_{i} f(x_i) \text{ ("Sum over all values of the index } i\text{")}$$

$$= \sum_{x} f(x) \text{ ("Sum over all possible values of } X\text{")}$$

8. Several summation signs can be used in one expression. Suppose the variable *Y* takes *n* values and *X* takes *m* values, and let $f(x,y)=x+y$. Then the *double summation* of this function is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} (x_i + y_j)$$

To evaluate such expressions work from the innermost sum outward. First set i=1 and sum over all values of j, and so on. That is,

To illustrate, let m = 2 and n = 3. Then

$$\begin{aligned}\sum_{i=1}^{2}\sum_{j=1}^{3} f\left(x_i, y_j\right) &= \sum_{i=1}^{2}\left[f\left(x_i, y_1\right) + f\left(x_i, y_2\right) + f\left(x_i, y_3\right)\right] \\ &= f\left(x_1, y_1\right) + f\left(x_1, y_2\right) + f\left(x_1, y_3\right) + \\ &\quad f\left(x_2, y_1\right) + f\left(x_2, y_2\right) + f\left(x_2, y_3\right)\end{aligned}$$

The *order* of summation does not matter, so

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j) = \sum_{j=1}^{n}\sum_{i=1}^{m} f(x_i, y_j)$$

2.3.2 The Mean of a Random Variable

- The expected value of a random variable X is the average value of the random variable in an infinite number of repetitions of the experiment (repeated samples); it is denoted $E[X]$.
- If X is a discrete random variable which can take the values x_1, $x_2,\ldots,x_n$ with probability density values $f(x_1)$, $f(x_2),\ldots,$ $f(x_n)$, the expected value of X is

$$\begin{aligned} E[X] &= x_1 f(x_1) + x_2 f(x_2) + \cdots + x_n f(x_n) \\ &= \sum_{i=1}^{n} x_i f(x_i) \qquad (2.3.1) \\ &= \sum_{x} x f(x) \end{aligned}$$

2.3.3 Expectation of a Function of a Random Variable

- If X is a discrete random variable and $g(X)$ is a function of it, then

$$E[g(X)] = \sum_x g(x) f(x) \qquad (2.3.2a)$$

However, $E[g(X)] \neq g[E(X)]$ in general.

- If X is a discrete random variable and $g(X) = g_1(X) + g_2(X)$, where $g_1(X)$ and $g_2(X)$ are functions of X, then

$$\begin{aligned} E[g(X)] &= \sum_x [g_1(x) + g_2(x)] f(x) \\ &= \sum_x g_1(x) f(x) + \sum_x g_2(x) f(x) \qquad (2.3.2b) \\ &= E[g_1(x)] + E[g_2(x)] \end{aligned}$$

- The expected value of a sum of functions of random variables, or the expected value of a sum of random variables, is always the sum of the expected values.

If c is a constant,

$$E[c] = c \qquad (2.3.3a)$$

If c is a constant and X is a random variable, then

$$E[cX] = cE[X] \tag{2.3.3b}$$

If a and c are constants then

$$E[a + cX] = a + cE[X] \tag{2.3.3c}$$

2.3.4 The Variance of a Random Variable

$$\text{var}(X) = \sigma^2 = E[g(X)] = E\left[X - E(X)\right]^2 = E[X^2] - [E(X)]^2 \tag{2.3.4}$$

Let a and c be constants, and let $Z = a + cX$. Then Z is a random variable and its variance is

$$\text{var}(a + cX) = E[(a + cX) - E(a + cX)]^2 = c^2\,\text{var}(X) \tag{2.3.5}$$

2.4 Using Joint Probability Density Functions

2.4.1 Marginal Probability Density Functions

- If X and Y are two discrete random variables then

$$\begin{aligned} f(x) &= \sum_y f(x,y) \quad \text{for each value } X \text{ can take} \\ f(y) &= \sum_x f(x,y) \quad \text{for each value } Y \text{ can take} \end{aligned} \qquad (2.4.1)$$

2.4.2 Conditional Probability Density Functions

$$f(x \mid y) = P[X = x \mid Y = y] = \frac{f(x,y)}{f(y)} \qquad (2.4.2)$$

2.4.3 Independent Random Variables

- If X and Y are independent random variables, then

$$f(x, y) = f(x) f(y) \qquad (2.4.3)$$

for each and every pair of values x and y. The converse is also true.

- If $X_1, \ldots, X_n$ are statistically independent the joint probability density function can be factored and written as

$$f(x_1, x_2, \cdots, x_n) = f_1(x_1) \cdot f_2(x_2) \cdot \ldots \cdot f_n(x_n) \qquad (2.4.4)$$

- If X and Y are independent random variables, then the conditional probability density function of X given that $Y=y$ is

$$f(x \mid y) = \frac{f(x,y)}{f(y)} = \frac{f(x)f(y)}{f(y)} = f(x) \qquad (2.4.5)$$

for each and every pair of values x and y. The converse is also true.

2.5 The Expected Value of a Function of Several Random Variables: Covariance and Correlation

- If X and Y are random variables, then their covariance is

$$\text{cov}(X,Y) = E[(X - E[X])(Y - E[Y])] \qquad (2.5.1)$$

- If X and Y are discrete random variables, $f(x,y)$ is their joint probability density function, and $g(X,Y)$ is a function of them, then

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y) \tag{2.5.2}$$

- If X and Y are discrete random variables and $f(x,y)$ is their joint probability density function, then

$$\begin{aligned} \operatorname{cov}(X,Y) &= E[(X - E[X])(Y - E[Y])] \\ &= \sum_x \sum_y [x - E(X)][y - E(Y)] f(x,y) \end{aligned} \tag{2.5.3}$$

- If X and Y are random variables then their *correlation* is

$$\rho = \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}} \tag{2.5.4}$$

2.5.1 The Mean of a Weighted Sum of Random Variables

$$E[aX + bY] = aE(X) + bE(Y) \tag{2.5.5}$$

- If X and Y are random variables, then
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$$E\left[X + Y\right] = E\left[X\right] + E\left[Y\right] \tag{2.5.6}$$

2.5.2 The Variance of a Weighted Sum of Random Variables

- If X, Y, and Z are random variables and a, b, and c are constants, then

$$\begin{aligned}\operatorname{var}[aX+bY+cZ] &= a^2\operatorname{var}[X]+b^2\operatorname{var}[Y]+c^2\operatorname{var}[Z]\\ &\quad +2ab\operatorname{cov}[X,Y]+2ac\operatorname{cov}[X,Z]+2bc\operatorname{cov}[Y,Z]\end{aligned} \tag{2.5.7}$$

- If X, Y, and Z are independent, or uncorrelated, random variables, then the covariance terms are zero and:

$$\operatorname{var}[aX+bY+cZ] = a^2\operatorname{var}[X]+b^2\operatorname{var}[Y]+c^2\operatorname{var}[Z] \tag{2.5.8}$$

- If X, Y, and Z are independent, or uncorrelated, random variables, and if $a = b = c = 1$, then

$$\operatorname{var}[X + Y + Z] = \operatorname{var}[X] + \operatorname{var}[Y] + \operatorname{var}[Z] \tag{2.5.9}$$

2.6 The Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(x-\beta)^2}{2\sigma^2}\right], \qquad -\infty < x < \infty \tag{2.6.1}$$

$$Z = \frac{X - \beta}{\sigma} \sim N(0,1)$$

- If $X \sim N(\beta,\ \sigma^2)$ and a is a constant, then

$$P[X \geq a] = P\left[\frac{X-\beta}{\sigma} \geq \frac{a-\beta}{\sigma}\right] = P\left[Z \geq \frac{a-\beta}{\sigma}\right] \tag{2.6.2}$$

- If $X \sim N(\beta,\ \sigma^2)$ and a and b are constants, then

$$P[a \leq X \leq b] = P\left[\frac{a-\beta}{\sigma} \leq \frac{X-\beta}{\sigma} \leq \frac{b-\beta}{\sigma}\right] = P\left[\frac{a-\beta}{\sigma} \leq Z \leq \frac{b-\beta}{\sigma}\right] \tag{2.6.3}$$

- If $X_1 \sim N\left(\beta_1, \sigma_1^2\right)$, $X_2 \sim N\left(\beta_2, \sigma_2^2\right)$, $X_3 \sim N\left(\beta_3, \sigma_3^2\right)$ and c_1, c_2, c_3 are constants, then

$$Z = c_1X_1 + c_2X_2 + c_3X_3 \sim N\left[E\left(Z\right), \mathrm{var}\left(Z\right)\right] \tag{2.6.4}$$