3. *Interval Estimation When the Population Variance σ^2 is Not Known*

(3.1) The *t*-distribution

The methods developed in Section 2 require that we know the population variance σ^2. The statistic that is the basis of interval estimation when σ^2 is known is the standardized normal *N*(0,1) random variable

$$Z = \frac{b-\beta}{\sigma/\sqrt{T}} \tag{3.1}$$

When σ^2 is unknown it is natural to replace it with its estimator

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{T}\left(y_i - \bar{y}\right)^2}{T-1} \tag{3.2}$$

However, when we do so

$$\frac{b-\beta}{\hat{\sigma}/\sqrt{T}} \tag{3.3}$$

is the ratio of two random variables *b* and $\hat{\sigma}^2$, and this ratio no longer has a standard normal distribution.

The correct probability distribution of (3.3) was worked out by W.S. Gossett, an employee of the Guiness Brewery, who, in 1919, published his work under the pseudonym "Student." Gossett called the statistic "*t*" and hence its distribution is called "Student's" *t*-distribution. The statistic is actually a clever combination of the *Z* statistic in (3.1) and an independent chi-square random variable

$$V = \frac{(T-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{(T-1)} \tag{3.4}$$

Specifically,

$$\begin{aligned} t &= \frac{Z}{\sqrt{V/(T-1)}} \\ &= \frac{\dfrac{b-\beta}{\sigma/\sqrt{T}}}{\sqrt{\dfrac{(T-1)\hat{\sigma}^2}{\sigma^2}\Big/(T-1)}} = \frac{\dfrac{b-\beta}{\sigma/\sqrt{T}}}{\dfrac{\hat{\sigma}}{\sigma}} \\ &= \frac{b-\beta}{\hat{\sigma}/\sqrt{T}} \sim t_{(T-1)} \end{aligned} \tag{3.5}$$

The notation $t_{(T-1)}$ denotes a *t*-distribution with *T*–1 “degrees of freedom.” The degrees of freedom parameter is important because it determines the shape of the *t*-distribution. In general, a *t*-random variable with *m* degrees of freedom is formed by dividing a standardized normal random variable by an independent chi-square with *m* degrees of freedom,

$$t = \frac{N(0,1)}{\sqrt{\chi^2_{(m)}/m}} \sim t_{(m)} \tag{3.6}$$

In Figure 2 the probability density functions for *N*(0,1) and a $t_{(3)}$ random variables are graphed.

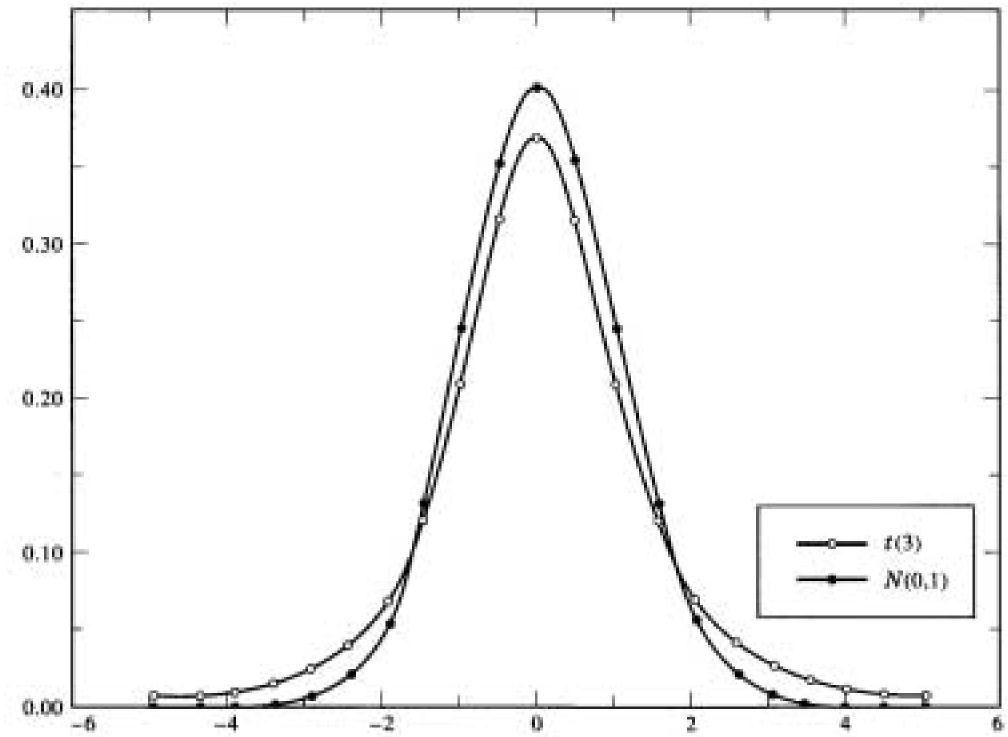

Figure 2 The standard normal and $t_{(3)}$ distributions

Note that the *t*-distribution is less “peaked” and there is more probability in the tails than for the standard normal. The probability density of a $t_{(m)}$ random variable is symmetric, with mean 0 and variance *m/*(*m*–2). As the degrees of freedom $m \to \infty$ the probability density function for the $t_{(m)}$ random variable approaches that of a standard normal *N*(0,1) random variable. Critical values for the *t*-distribution are contained in Table 2 in *UE/2*.

(3.2) Interval estimation

When σ^2 is unknown we can use (3.5) as a basis for interval estimation. If

$$t = \frac{b-\beta}{\hat{\sigma}/\sqrt{T}} \sim t_{(T-1)} \tag{3.7}$$

then,

$$P\left[-t_c \le \frac{b-\beta}{\hat{\sigma}/\sqrt{T}} \le t_c\right] = 1-\alpha \tag{3.8}$$

where t_c is a critical value from the *t*-distribution. Then, following the same steps as in section 2,

$$P\left[b - t_c \frac{\hat{\sigma}}{\sqrt{T}} \le \beta \le b + t_c \frac{\hat{\sigma}}{\sqrt{T}}\right] = 1-\alpha \tag{3.9}$$

The 100(1−α)% interval estimator for β is

$$b \pm t_c \frac{\hat{\sigma}}{\sqrt{T}} \tag{3.10}$$

Unlike the interval estimator for the known σ^2 case in (2.11), the interval in (3.10) has center and width that vary from sample to sample.

(3.3) A Simulation Experiment

As we did in Section 2.1 we use a computer simulation, or Monte Carlo experiment, to illustrate the sampling properties of the interval estimator in (3.10). We create 10 samples of size T = 30 from a normal population with mean β = 10 and variance $\sigma^2 = 10$. Table 3 contains the results of the experiment.

Table 3 Interval Estimates Using (3.10) from 10 samples

Sample	b	$\hat{\sigma}^2$	Lower bound	Upper bound
1	10.206	9.199	9.073	11.339
2	9.828	6.876	8.849	10.807
3	11.194	10.330	9.994	12.394
4	8.822	9.868	7.649	9.995
5	10.434	7.985	9.379	11.489
6	8.855	6.231	7.923	9.787
7	10.511	7.333	9.500	11.523
8	9.212	14.686	7.781	10.643
9	10.464	10.414	9.259	11.669
10	10.142	17.690	8.571	11.712

The estimates *b* are the same as in Table 2. The estimates $\hat{\sigma}^2$ vary about the true value $\sigma^2 = 10$. Of these 10 intervals, those for samples 4 and 6 do not contain the true parameter β = 10. Nevertheless, in a large number of samples 95% of them will contain the true population mean β.

(3.4) An Empirical Example

In *Statistical Inference I* we introduced the empirical problem faced by an airplane seat design engineer. Given a random sample of size T = 50 we estimated the mean U.S. hip width to be b = 17.158 inches. Furthermore we estimated the population variance to be

$$\hat{\sigma}^2 = 3.267 \tag{3.11}$$

Thus $\hat{\sigma} = 1.807$. To construct a 95% interval estimate we use (3.10),

$$b \pm t_c \frac{\hat{\sigma}}{\sqrt{T}} = 17.158 \pm 2.01 \frac{1.807}{\sqrt{50}} = [16.644, 17.672] \tag{3.12}$$

We estimate that the population mean hip size falls between 16.644 and 17.672 inches. While we do not know if this interval contains the true population mean hip size for sure, we know that the procedure used to create the interval "works" 95% of the time, thus we would be surprised if the interval did not contain the true population value β.