Dynamic Models, Autocorrelation and Forecasting

Chapter 9

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Chapter 9: Dynamic Models, Autocorrelation and Forecasting

- 9.1 Introduction
- 9.2 Lags in the Error Term: Autocorrelation
- 9.3 Estimating an AR(1) Error Model
- 9.4 Testing for Autocorrelation
- 9.5 An Introduction to Forecasting: Autoregressive Models
- 9.6 Finite Distributed Lags
- 9.7 Autoregressive Distributed Lag Models

9.1 Introduction

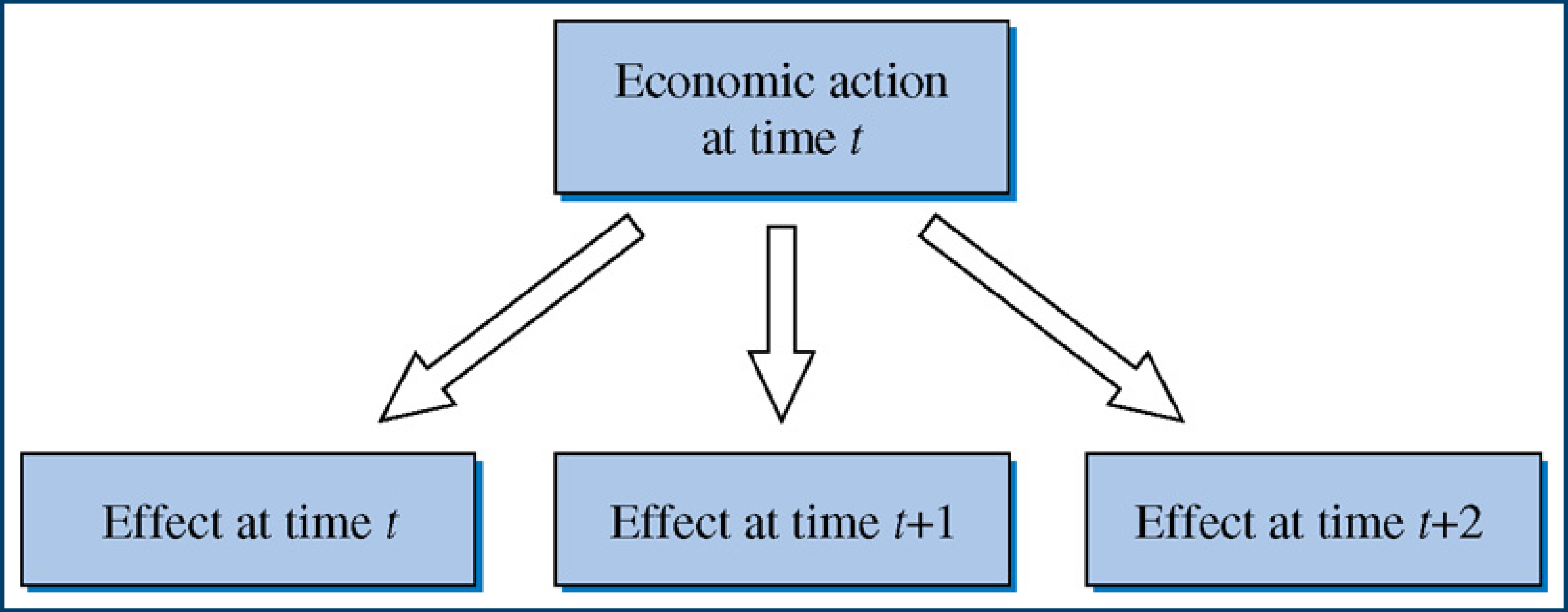

Figure 9.1

9.1 Introduction

$$y_t = f(x_t, x_{t-1}, x_{t-2}, \ldots) \qquad (9.1)$$

$$y_t = f(y_{t-1}, x_t) \qquad (9.2)$$

$$y_t = f(x_t) + e_t \qquad e_t = f(e_{t-1}) \qquad (9.3)$$

9.1 Introduction

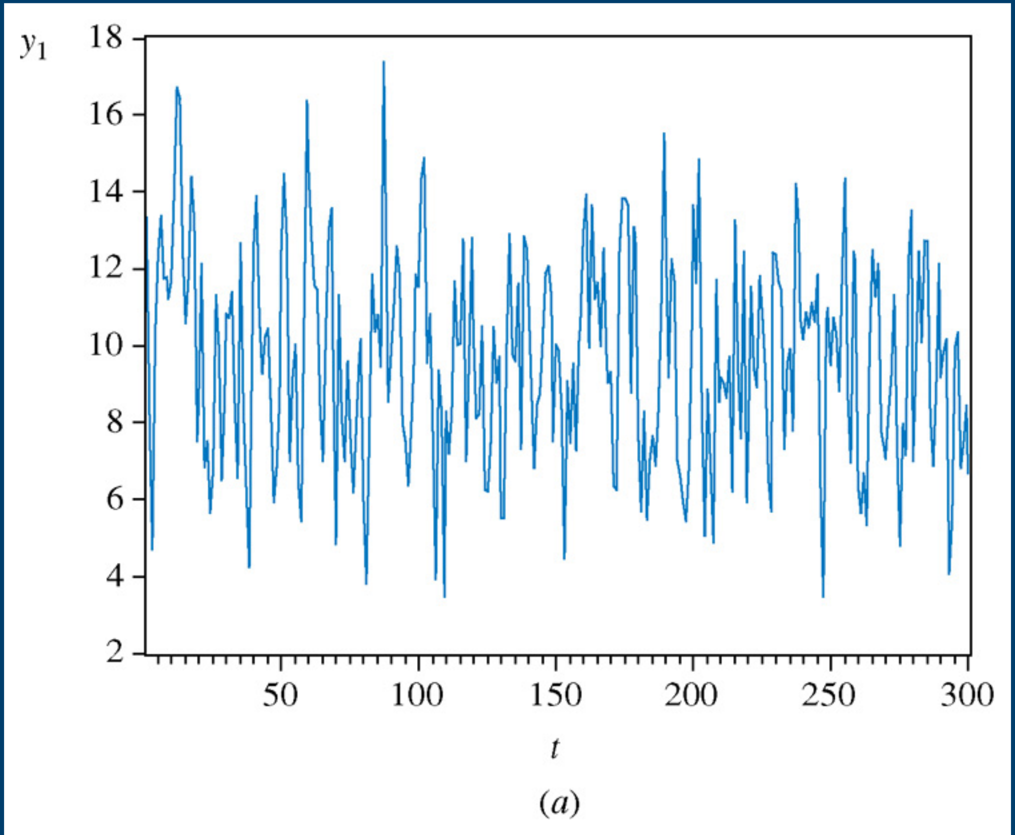

Figure 9.2(a) Time Series of a Stationary Variable

9.1 Introduction

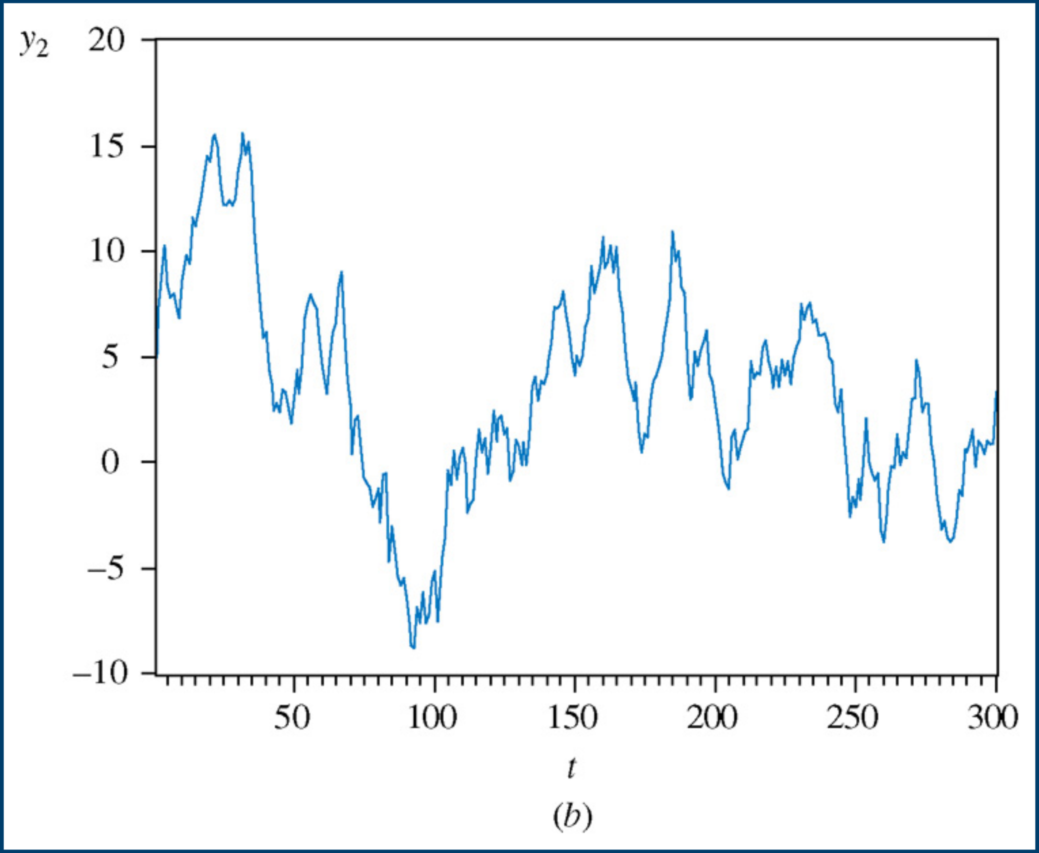

Figure 9.2(b) Time Series of a Nonstationary Variable that is 'Slow Turning' or 'Wandering'

9.1 Introduction

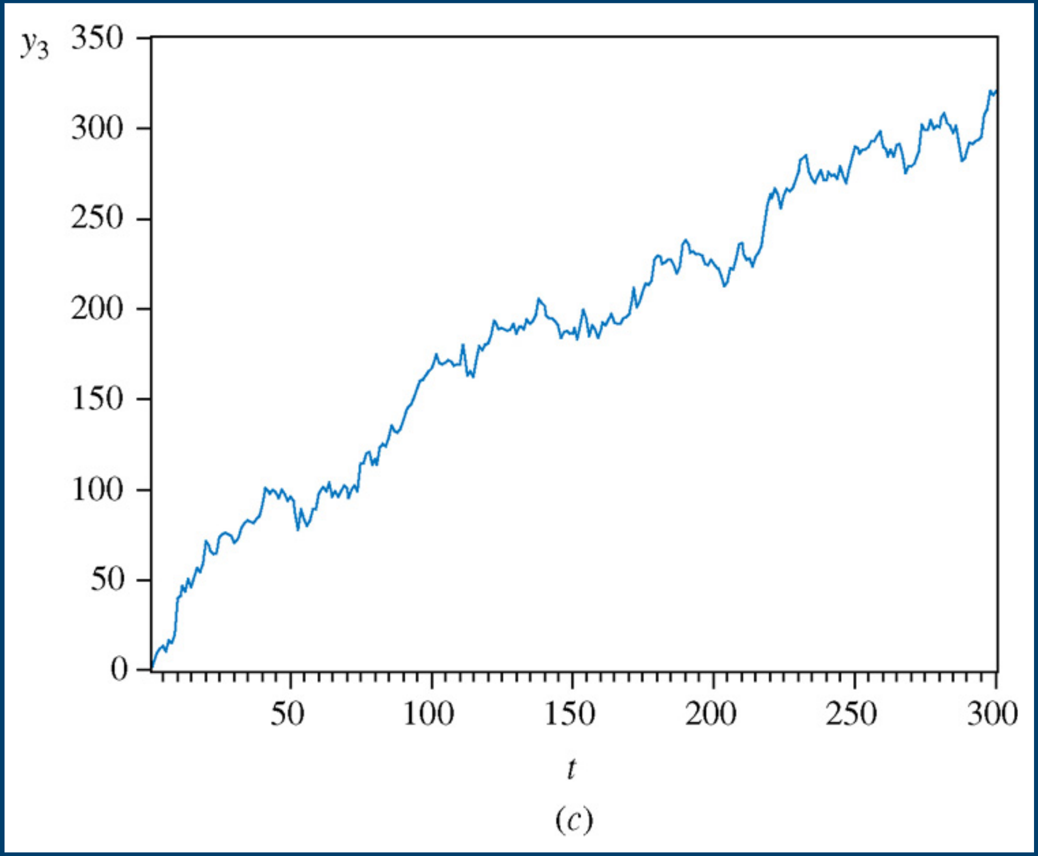

Figure 9.2(c) Time Series of a Nonstationary Variable that 'Trends'

9.2 Lags in the Error Term: Autocorrelation

9.2.1 Area Response Model for Sugar Cane

$$\ln(A) = \beta_1 + \beta_2 \ln(P)$$

$$\ln(A_t) = \beta_1 + \beta_2 \ln(P_t) + e_t \quad (9.4)$$

$$y_t = \beta_1 + \beta_2 x_t + e_t \quad (9.5)$$

$$e_t = \rho e_{t-1} + v_t \quad (9.6)$$

9.2.2 First-Order Autoregressive Errors

$$y_t = \beta_1 + \beta_2 x_t + e_t \quad (9.7)$$

$$e_t = \rho e_{t-1} + v_t \quad (9.8)$$

$$E(v_t) = 0 \quad \text{var}(v_t) = \sigma_v^2 \quad \text{cov}(v_t, v_s) = 0 \quad \text{for } t \neq s \quad (9.9)$$

$$-1 < \rho < 1 \quad (9.10)$$

9.2.2 First-Order Autoregressive Errors

$$E(e_t) = 0 \tag{9.11}$$

$$\text{var}(e_t) = \sigma_e^2 = \frac{\sigma_v^2}{1-\rho^2} \tag{9.12}$$

$$\text{cov}\left(e_t, e_{t-k}\right) = \sigma_e^2 \rho^k \qquad k > 0 \tag{9.13}$$

9.2.2 First-Order Autoregressive Errors

$$\operatorname{corr}(e_t, e_{t-k}) = \frac{\operatorname{cov}(e_t, e_{t-k})}{\sqrt{\operatorname{var}(e_t)\operatorname{var}(e_{t-k})}} = \frac{\operatorname{cov}(e_t, e_{t-k})}{\operatorname{var}(e_t)} = \frac{\sigma_e^2 \rho^k}{\sigma_e^2} = \rho^k \qquad (9.14)$$

$$\operatorname{corr}(e_t, e_{t-1}) = \rho \qquad (9.15)$$

$$\begin{aligned} \hat{y}_t &= 3.893 + .776\, x_t \\ \text{(se)} \quad & (.061) \quad (.277) \end{aligned} \qquad (9.16)$$

9.2.2 First-Order Autoregressive Errors

Table 9.1 Least Squares Residuals for the Sugarcane Example

Time	$\hat{e}_t$	Time	$\hat{e}_t$	Time	$\hat{e}_t$	Time	$\hat{e}_t$
1	−0.303	10	−0.254	19	−0.036	27	−0.651
2	0.254	11	−0.145	20	0.361	28	−0.218
3	0.182	12	0.091	21	−0.138	29	0.137
4	0.503	13	0.304	22	0.017	30	0.121
5	0.275	14	0.656	23	0.336	31	−0.040
6	−0.115	15	0.134	24	−0.175	32	−0.048
7	−0.437	16	−0.059	25	−0.517	33	0.183
8	−0.423	17	0.435	26	−0.137	34	0.184
9	−0.367	18	−0.106				

9.2.2 First-Order Autoregressive Errors

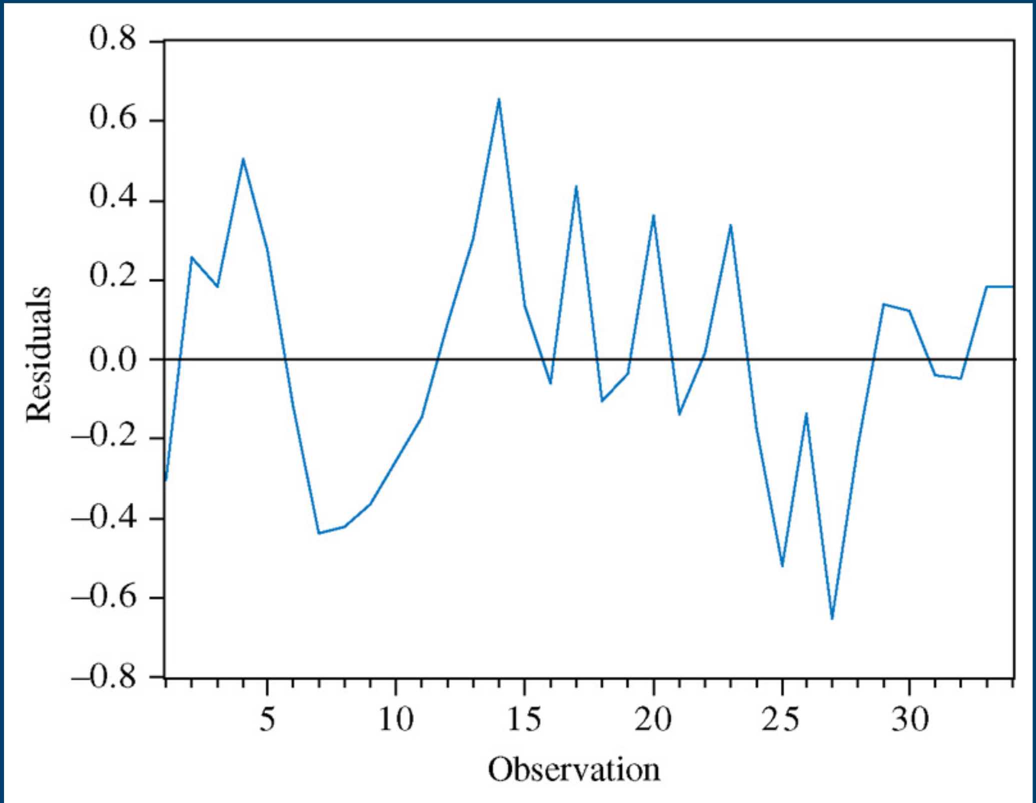

Figure 9.3 Least Squares Residuals Plotted Against Time

9.2.2 First-Order Autoregressive Errors

$$r_{xy} = \frac{\widehat{\text{cov}}(x_t, y_t)}{\sqrt{\widehat{\text{var}}(x_t)\widehat{\text{var}}(y_t)}} = \frac{\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y})}{\sqrt{\sum_{t=1}^{T}(x_t - \bar{x})^2 \sum_{t=1}^{T}(y_t - \bar{y})^2}} \qquad (9.17)$$

$$r_1 = \frac{\widehat{\text{cov}}(e_t, e_{t-1})}{\sqrt{\widehat{\text{var}}(e_t)}} = \frac{\sum_{t=2}^{T}\hat{e}_t\hat{e}_{t-1}}{\sum_{t=2}^{T}\hat{e}_{t-1}^2} \qquad (9.18)$$

9.3 Estimating an AR(1) Error Model

The existence of AR(1) errors implies:

- The least squares estimator is still a linear and unbiased estimator, but it is no longer best. There is another estimator with a smaller variance.
- The standard errors usually computed for the least squares estimator are incorrect. Confidence intervals and hypothesis tests that use these standard errors may be misleading.

9.3 Estimating an AR(1) Error Model

Sugar cane example

The two sets of standard errors, along with the estimated equation are:

$$\hat{y}_t = 3.893 + .776\,x_t$$

$$(.061)\;(.277) \qquad \text{'incorrect' se's}$$

$$(.062)\;(.378) \qquad \text{'correct' se's}$$

The 95% confidence intervals for β_2 are:

$$(.211, 1.340) \qquad \text{(incorrect)}$$

$$(.006, 1.546) \qquad \text{(correct)}$$

9.3.2 Nonlinear Least Squares Estimation

$$y_t = \beta_1 + \beta_2 x_t + e_t \tag{9.19}$$

$$e_t = \rho e_{t-1} + v_t \tag{9.20}$$

$$y_t = \beta_1 + \beta_2 x_t + \rho e_{t-1} + v_t \tag{9.21}$$

$$e_{t-1} = y_{t-1} - \beta_1 - \beta_2 x_{t-1} \tag{9.22}$$

9.3.2 Nonlinear Least Squares Estimation

$$\rho e_{t-1} = \rho y_{t-1} - \rho\beta_1 - \rho\beta_2 x_{t-1} \quad (9.23)$$

$$y_t = \beta_1(1-\rho) + \beta_2 x_t + \rho y_{t-1} - \rho\beta_2 x_{t-1} + v_t \quad (9.24)$$

$$\widehat{\ln(A_t)} = 3.899 + .888\ln(P_t) \qquad \hat{e}_t = .422 e_{t-1} + v_t \quad (9.25)$$

$$\text{(se)} \quad (.092) \quad (.259) \qquad\qquad (.166)$$

9.3.2a Generalized Least Squares Estimation

It can be shown that nonlinear least squares estimation of (9.24) is equivalent to using an iterative generalized least squares estimator called the Cochrane-Orcutt procedure. Details are provided in Appendix 9A.

9.3.3 Estimating a More General Model

$$y_t = \beta_1(1-\rho) + \beta_2 x_t - \rho\beta_2 x_{t-1} + \rho y_{t-1} + v_t \quad (9.26)$$

$$y_t = \delta + \delta_0 x_t + \delta_1 x_{t-1} + \theta_1 y_{t-1} + v_t \quad (9.27)$$

$$\delta = \beta_1(1-\rho) \qquad \delta_0 = \beta_2 \qquad \delta_1 = -\rho\beta_2 \qquad \theta_1 = \rho$$

$$\hat{y}_t = 2.366 + .777 x_t - .611 x_{t-1} + .404 y_{t-1} \quad (9.28)$$

$$\text{(se)} \quad (.656) \quad (.280) \quad (.297) \quad (.167)$$

9.4 Testing for Autocorrelation

9.4.1 Residual Correlogram

$$H_0 : \rho = 0 \qquad H_1 : \rho \neq 0$$

$$z = \sqrt{T} r_1 \sim N(0,1) \tag{9.29}$$

$$z = \sqrt{34} \times .404 = 2.36 \geq 1.96 \tag{9.30}$$

9.4 Testing for Autocorrelation

9.4.1 Residual Correlogram

$$r_1 \geq \frac{1.96}{\sqrt{T}} \quad \text{or} \quad r_1 \leq -\frac{1.96}{\sqrt{T}}$$

$$r_k \geq \frac{1.96}{\sqrt{T}} \quad \text{or} \quad r_k \leq -\frac{1.96}{\sqrt{T}} \qquad (9.31)$$

$$\rho_k = \frac{\text{cov}(e_t, e_{t-k})}{\text{var}(e_t)} = \frac{E(e_t e_{t-k})}{E(e_t^2)} \qquad (9.32)$$

9.4.1 Residual Correlogram

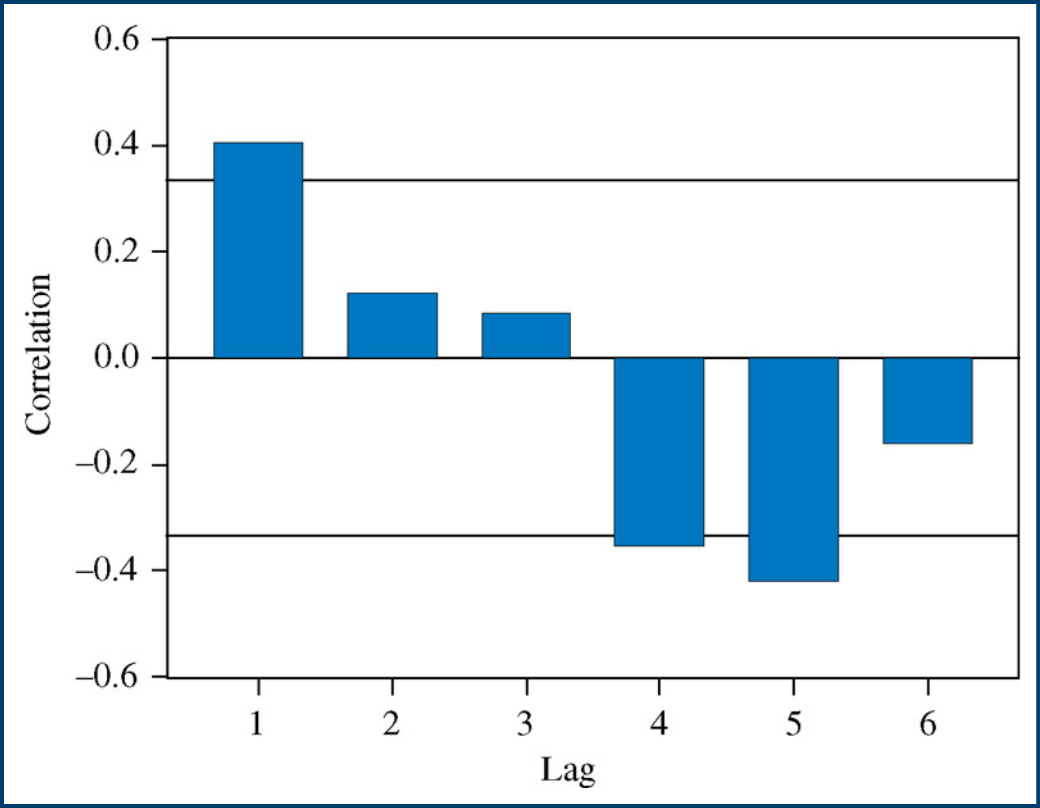

Figure 9.4 Correlogram for Least Squares Residuals from Sugar Cane Example

9.4.1 Residual Correlogram

$$y_t = \beta_1 + \beta_2 x_t + e_t$$

$$y_t = \beta_1(1-\rho) + \beta_2 x_t + \rho y_{t-1} - \rho\beta_2 x_{t-1} + v_t$$

9.4.1 Residual Correlogram

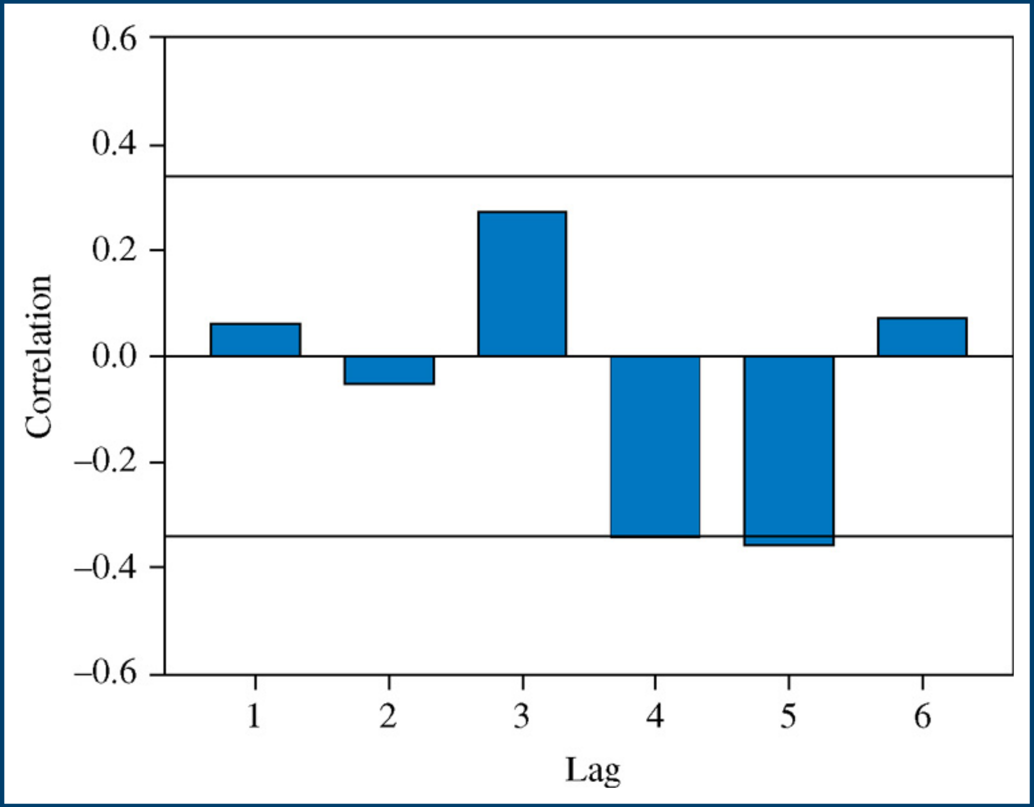

Figure 9.5 Correlogram for Nonlinear Least Squares Residuals from Sugar Cane Example

9.4.2 A Lagrange Multiplier Test

$$y_t = \beta_1 + \beta_2 x_t + \rho e_{t-1} + v_t \qquad (9.33)$$

$t = 2.439 \qquad F = 5.949 \qquad p\text{-value} = .021$

$$y_t = \beta_1 + \beta_2 x_t + \rho \hat{e}_{t-1} + v_t \qquad (9.34)$$

$$b_1 + b_2 x_t + \hat{e}_t = \beta_1 + \beta_2 x_t + \rho e_{t-1} + \hat{v}_t$$

9.4.2 A Lagrange Multiplier Test

$$\hat{e}_t = (\beta_1 - b_1) + (\beta_2 - b_2)x_t + \rho e_{t-1} + \hat{v}_t$$

$$= \gamma_1 + \gamma_2 x_t + \rho \hat{e}_{t-1} + v_t \qquad (9.35)$$

$$LM = T \times R^2 = 34 \times .16101 = 5.474$$

9.5 An Introduction to Forecasting: Autoregressive Models

$$y_t = \delta + \theta_1 y_{t-1} + \theta_2 y_{t-2} + \cdots + \theta_p y_{t-p} + v_t \qquad (9.36)$$

$$y_t = \left(\ln(CPI_t) - \ln(CPI_{t-1})\right) \times 100 \approx \left(\frac{CPI_t - CPI_{t-1}}{CPI_{t-1}}\right) \times 100$$

$$\underset{(se)}{\hat{INFLN}_t} = \underset{(.0253)}{.1883} + \underset{(.0615)}{.3733}\, INFLN_{t-1} - \underset{(.0645)}{.2179}\, INFLN_{t-2} + \underset{(.0613)}{.1013}\, INFLN_{t-3} \qquad (9.37)$$

9.5 An Introduction to Forecasting: Autoregressive Models

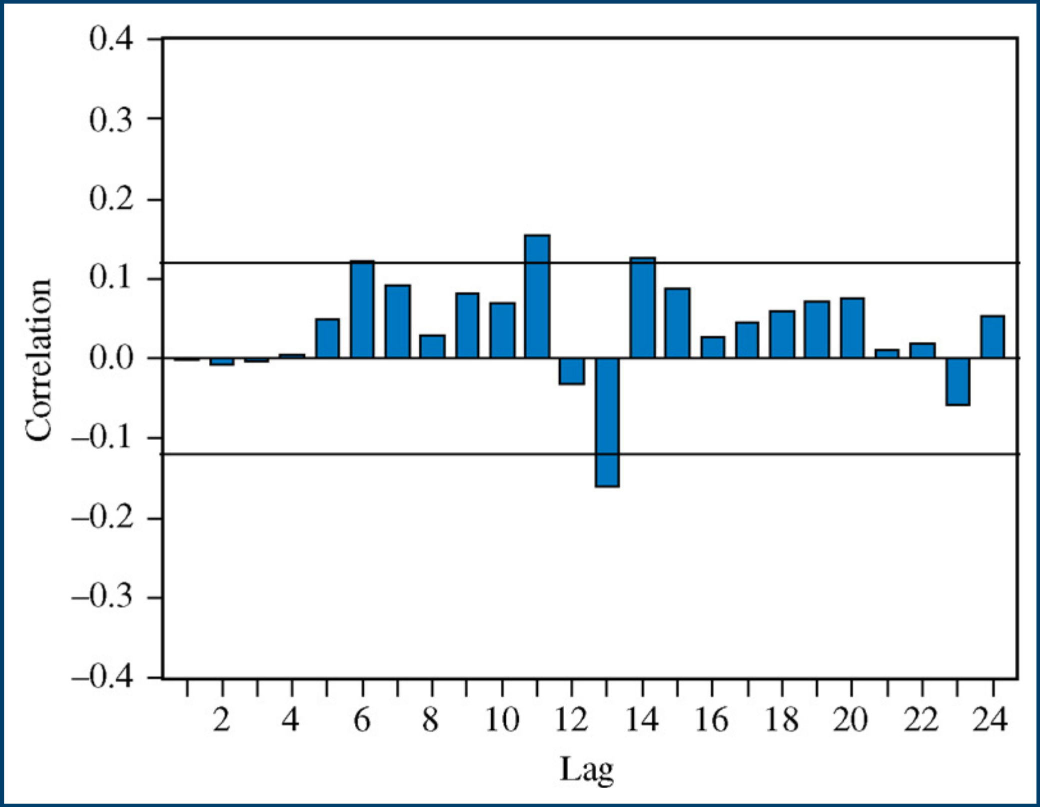

Figure 9.6 Correlogram for Least Squares Residuals from AR(3) Model for Inflation

9.5 An Introduction to Forecasting: Autoregressive Models

$$y_t = \delta + \theta_1 y_{t-1} + \theta_2 y_{t-2} + \theta_3 y_{t-3} + v_t \qquad (9.38)$$

$$y_{T+1} = \delta + \theta_1 y_T + \theta_2 y_{T-1} + \theta_3 y_{T-2} + v_{T+1}$$

$$\begin{aligned} \hat{y}_{T+1} &= \hat{\delta} + \theta_1 y_T + \hat{\theta}_2 y_{T-1} + \theta_3 y_{T-2} \\ &= .1883 + .3733 \times .4468 - .2179 \times .5988 + .1013 \times .3510 \\ &= .2602 \end{aligned}$$

9.5 An Introduction to Forecasting: Autoregressive Models

$$\hat{y}_{T+2} = \hat{\delta} + \theta_1 y_{T+1} + \hat{\theta}_2 y_T + \theta_3 y_{T-1}$$

$$= .1883 + .3733 \times .2602 - .2179 \times .4468 + .1013 \times .5988 \quad (9.39)$$

$$= .2487$$

$$u_1 = y_{T+1} - \hat{y}_{T+1} = (\delta - \hat{\delta}) + (\theta_1 - \theta_1) y_T + (\theta_2 - \hat{\theta}_2) y_{T-1} + (\theta_3 - \theta_3) y_{T-2} + v_{T+1}$$

9.5 An Introduction to Forecasting: Autoregressive Models

Table 9.2 Forecasts and Forecast Intervals for Inflation Rate

Month	Forecast $\hat{y}_{T+j}$	Standard error of forecast error $(\hat{\sigma}_j)$	Forecast interval $(\hat{y}_{T+j} - 1.969\,\hat{\sigma}_j, \hat{y}_{T+j} + 1.969\,\hat{\sigma}_j)$
Jun 06 ($j = 1$)	0.2602	0.1972	(−0.1282, 0.6485)
Jul 06 ($j = 2$)	0.2487	0.2105	(−0.1658, 0.6633)
Aug 06 ($j = 3$)	0.2697	0.2111	(−0.1460, 0.6854)

9.5 An Introduction to Forecasting: Autoregressive Models

$$u_1 = v_{T+1} \quad (9.40)$$

$$u_2 = \theta_1(y_{T+1} - \hat{y}_{T+1}) + v_{T+2} = \theta_1 u_1 + v_{T+2} = \theta_1 v_{T+1} + v_{T+2} \quad (9.41)$$

$$u_3 = \theta_1 u_2 + \theta_2 u_1 + v_{T+3} = (\theta_1^2 + \theta_2) v_{T+1} + \theta_1 v_{T+2} + v_{T+3} \quad (9.42)$$

9.5 An Introduction to Forecasting: Autoregressive Models

$$\sigma_1^2 = \text{var}(u_1) = \sigma_v^2$$

$$\sigma_2^2 = \text{var}(u_2) = \sigma_v^2(1+\theta_1^2)$$

$$\sigma_3^2 = \text{var}(u_3) = \sigma_v^2[(\theta_1^2+\theta_2)^2+\theta_1^2+1]$$

$$\left(\hat{y}_{T+j} - 1.96\times\sigma_j\,,\ \hat{y}_{T+j} + 1.96\times\sigma_j\right) \qquad (9.43)$$

9.6 Finite Distributed Lags

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \cdots + \beta_q x_{t-q} + v_t, \quad t = q+1, \ldots, T \qquad (9.44)$$

$$\frac{\partial E(y_t)}{\partial x_{t-s}} = \beta_s$$

$$x_t = \left(\ln(WAGE_t) - \ln(WAGE_{t-1})\right) \times 100 \approx \left(\frac{WAGE_t - WAGE_{t-1}}{WAGE_{t-1}}\right) \times 100$$

9.6 Finite Distributed Lags

Table 9.3 Least Squares Estimates for Finite Distributed Lag Model

Variable	Coefficient	Std. Error	t-value	p-value
Constant	0.1219	0.0487	2.505	0.013
x_t	0.1561	0.0885	1.764	0.079
x_{t-1}	0.1075	0.0851	1.264	0.207
x_{t-2}	0.0495	0.0853	0.580	0.562
x_{t-3}	0.1990	0.0879	2.264	0.024

9.6 Finite Distributed Lags

Table 9.4 **Multipliers for Inflation Example**

Lag	Multipliers	
	Delay	Interim
0	0.1561	0.1561
1	0.1075	0.2636
2	0.0495	0.3131
3	0.1990	0.5121

9.7 Autoregressive Distributed Lag Models

$$y_t = \delta + \delta_0 x_t + \delta_1 x_{t-1} + \cdots + \delta_q x_{t-q} + \theta_1 y_{t-1} + \cdots + \theta_p y_{t-p} + v_t \quad (9.45)$$

$$\begin{aligned} y_t &= \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \beta_3 x_{t-3} + \cdots + e_t \\ &= \alpha + \sum_{s=0}^{\infty} \beta_s x_{t-s} + e_t \end{aligned} \quad (9.46)$$

9.7 Autoregressive Distributed Lag Models

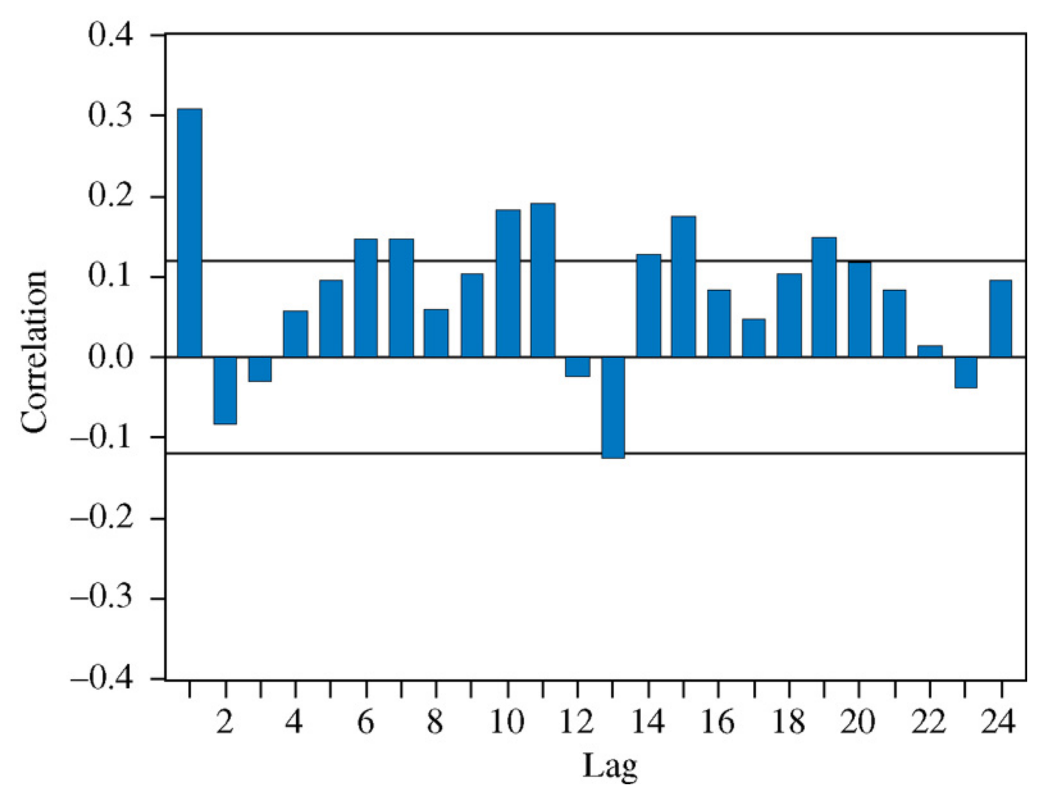

Figure 9.7 Correlogram for Least Squares Residuals from Finite Distributed Lag Model

9.7 Autoregressive Distributed Lag Models

$$\widehat{INFLN}_t = .0989 + .1149\,PCWAGE_t + .0377\,PCWAGE_{t-1} + .0593\,PCWAGE_{t-2}$$
$$\text{(se)} \quad (.0288) \quad (.0761) \quad (.0812) \quad (.0812)$$

$$+ .2361\,PCWAGE_{t-3} + .3536\,INFLN_{t-1} - .1976\,INFLN_{t-2}$$
$$(.0829) \quad (.0604) \quad (.0604)$$

(9.47)

9.7 Autoregressive Distributed Lag Models

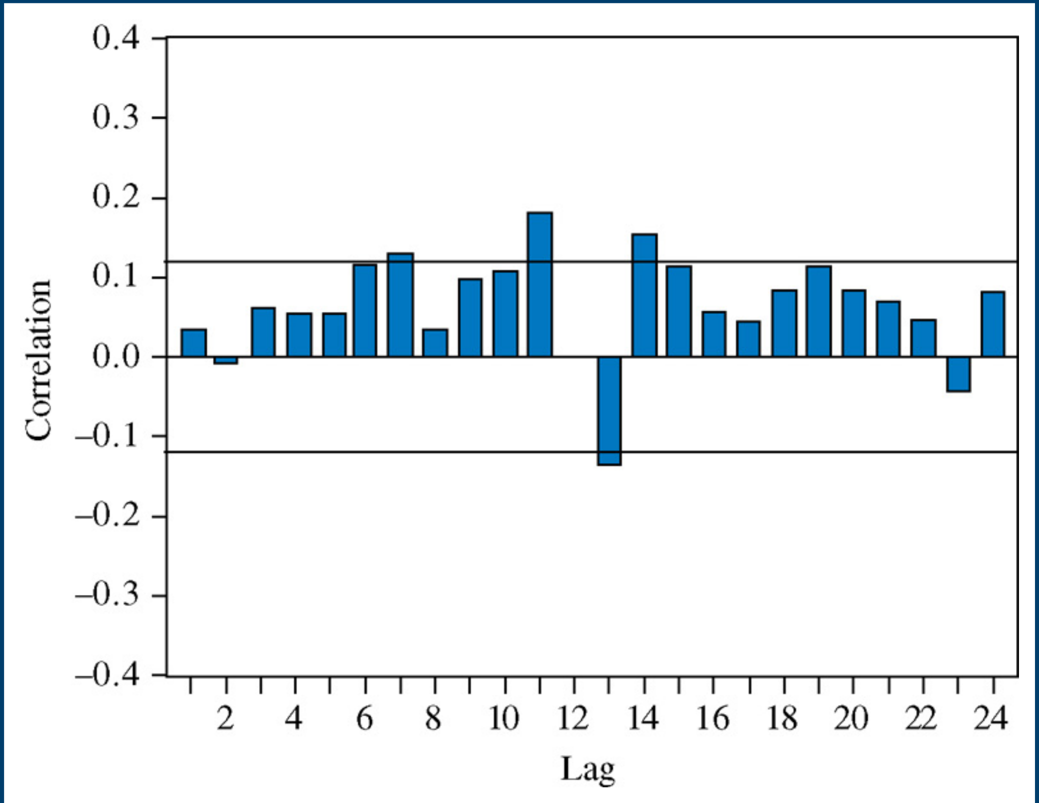

Figure 9.8 Correlogram for Least Squares Residuals from Autoregressive Distributed Lag Model

9.7 Autoregressive Distributed Lag Models

$$y_t = \delta + \delta_0 x_t + \delta_1 x_{t-1} + \delta_2 x_{t-2} + \delta_3 x_{t-3} + \theta_1 y_{t-1} + \theta_2 y_{t-2} + v_t$$

$$\hat{\beta}_0 = \delta_0 = .1149$$

$$\hat{\beta}_1 = \theta_1 \hat{\beta}_0 + \delta_1 = .3536 \times .1149 + .0377 = .0784$$

$$\hat{\beta}_2 = \theta_1 \hat{\beta}_1 + \theta_2 \hat{\beta}_0 + \delta_2 = .0643$$

$$\hat{\beta}_3 = \theta_1 \hat{\beta}_2 + \theta_2 \hat{\beta}_1 + \delta_3 = .2434$$

$$\hat{\beta}_4 = \theta_1 \hat{\beta}_3 + \theta_2 \hat{\beta}_2 = .0734$$

9.7 Autoregressive Distributed Lag Models

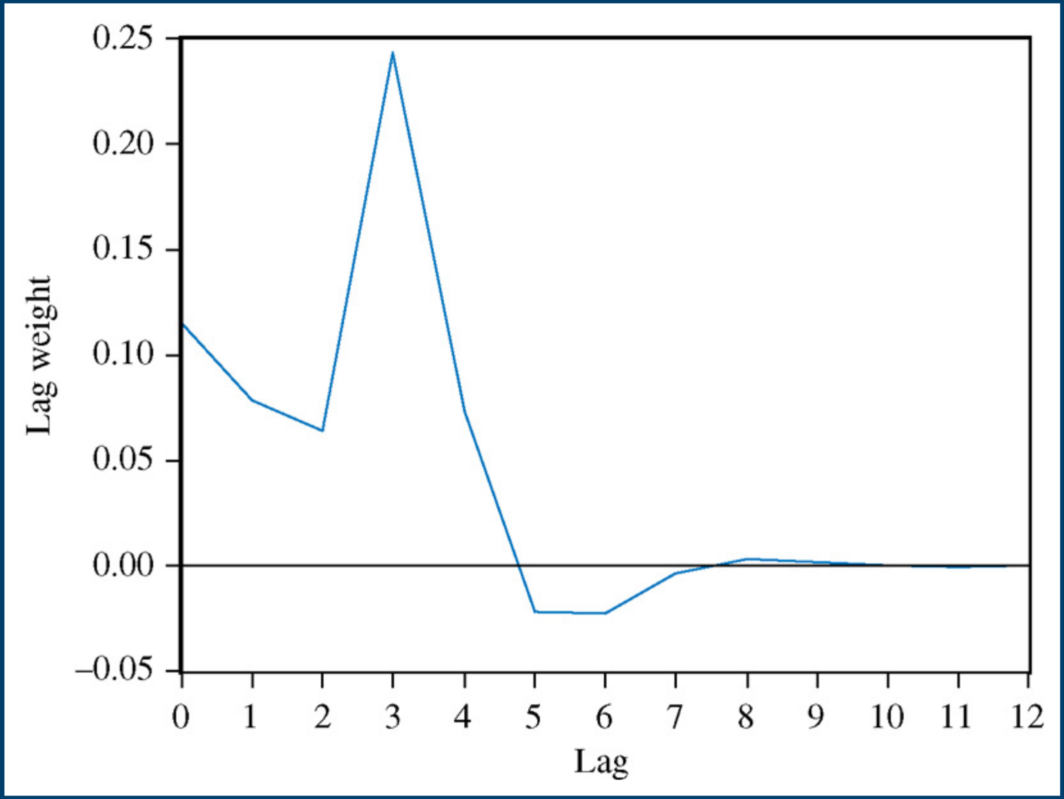

Figure 9.9 Distributed Lag Weights for Autoregressive Distributed Lag Model

Keywords

- autocorrelation
- autoregressive distributed lag models
- autoregressive error
- autoregressive model
- correlogram
- delay multiplier
- distributed lag weight
- dynamic models
- finite distributed lag
- forecast error
- forecasting
- HAC standard errors
- impact multiplier
- infinite distributed lag
- interim multiplier
- lag length
- lagged dependent variable
- *LM* test
- nonlinear least squares
- sample autocorrelation function
- standard error of forecast error
- total multiplier
- form of *LM* test

Chapter 9 Appendices

- Appendix 9A Generalized Least Squares Estimation
- Appendix 9B The Durbin Watson Test
- Appendix 9C Deriving ARDL Lag Weights
- Appendix 9D Forecasting: Exponential Smoothing

Appendix 9A
Generalized Least Squares Estimation

$$y_t = \beta_1 + \beta_2 x_t + e_t \qquad e_t = \rho e_{t-1} + v_t$$

$$y_t = \beta_1 + \beta_2 x_t + \rho y_{t-1} - \rho\beta_1 - \rho\beta_2 x_{t-1} + v_t \qquad (9A.1)$$

$$y_t - \rho y_{t-1} = \beta_1 (1-\rho) + \beta_2 (x_t - \rho x_{t-1}) + v_t \qquad (9A.2)$$

$$y_t^* = y_t - \rho y_{t-1} \qquad x_{t2}^* = x_t - \rho x_{t-1} \qquad x_{t1}^* = 1 - \rho$$

Appendix 9A
Generalized Least Squares Estimation

$$y_t^* = x_{t1}^* \beta_1 + x_{t2}^* \beta_2 + v_t \qquad (9A.3)$$

$$y_t - \beta_1 - \beta_2 x_t = \rho(y_{t-1} - \beta_1 - \beta_2 x_{t-1}) + v_t \qquad (9A.4)$$

Appendix 9A
Generalized Least Squares Estimation

$$y_1 = \beta_1 + x_1\beta_2 + e_1$$

$$\sqrt{1-\rho^2}\,y_1 = \sqrt{1-\rho^2}\,\beta_1 + \sqrt{1-\rho^2}\,x_1\beta_2 + \sqrt{1-\rho^2}\,e_1$$

$$y_1^* = x_{11}^*\beta_1 + x_{12}^*\beta_2 + e_1^* \qquad (9A.5)$$

$$y_1^* = \sqrt{1-\rho^2}\,y_1 \qquad x_{11}^* = \sqrt{1-\rho^2}$$

$$x_{12}^* = \sqrt{1-\rho^2}\,x_1 \qquad e_1^* = \sqrt{1-\rho^2}\,e_1 \qquad (9A.6)$$

Appendix 9A
Generalized Least Squares Estimation

$$\operatorname{var}(e_1^*) = (1-\rho^2)\operatorname{var}(e_1) = (1-\rho^2)\frac{\sigma_v^2}{1-\rho^2} = \sigma_v^2$$

Appendix 9B
The Durbin-Watson Test

$$H_0 : \rho = 0 \qquad H_1 : \rho > 0$$

$$d = \frac{\sum_{t=2}^{T} \left(\hat{e}_t - e_{t-1} \right)^2}{\sum_{t=1}^{T} \hat{e}_t^2} \qquad (9B.1)$$

Appendix 9B
The Durbin-Watson Test

$$d = \frac{\sum_{t=2}^{T} \hat{e}_t^2 + \sum_{t=2}^{T} \hat{e}_{t-1}^2 - 2\sum_{t=2}^{T} \hat{e}_t \hat{e}_{t-1}}{\sum_{t=1}^{T} \hat{e}_t^2}$$

$$= \frac{\sum_{t=2}^{T} \hat{e}_t^2}{\sum_{t=1}^{T} \hat{e}_t^2} + \frac{\sum_{t=2}^{T} \hat{e}_{t-1}^2}{\sum_{t=1}^{T} \hat{e}_t^2} - 2\,\frac{\sum_{t=2}^{T} \hat{e}_t \hat{e}_{t-1}}{\sum_{t=1}^{T} \hat{e}_t^2} \qquad (9B.2)$$

$$\approx 1 + 1 - 2r_1$$

Appendix 9B
The Durbin-Watson Test

$$d \approx 2\left(1 - r_1\right) \qquad (9B.3)$$

$$d \le d_c$$

Appendix 9B
The Durbin-Watson Test

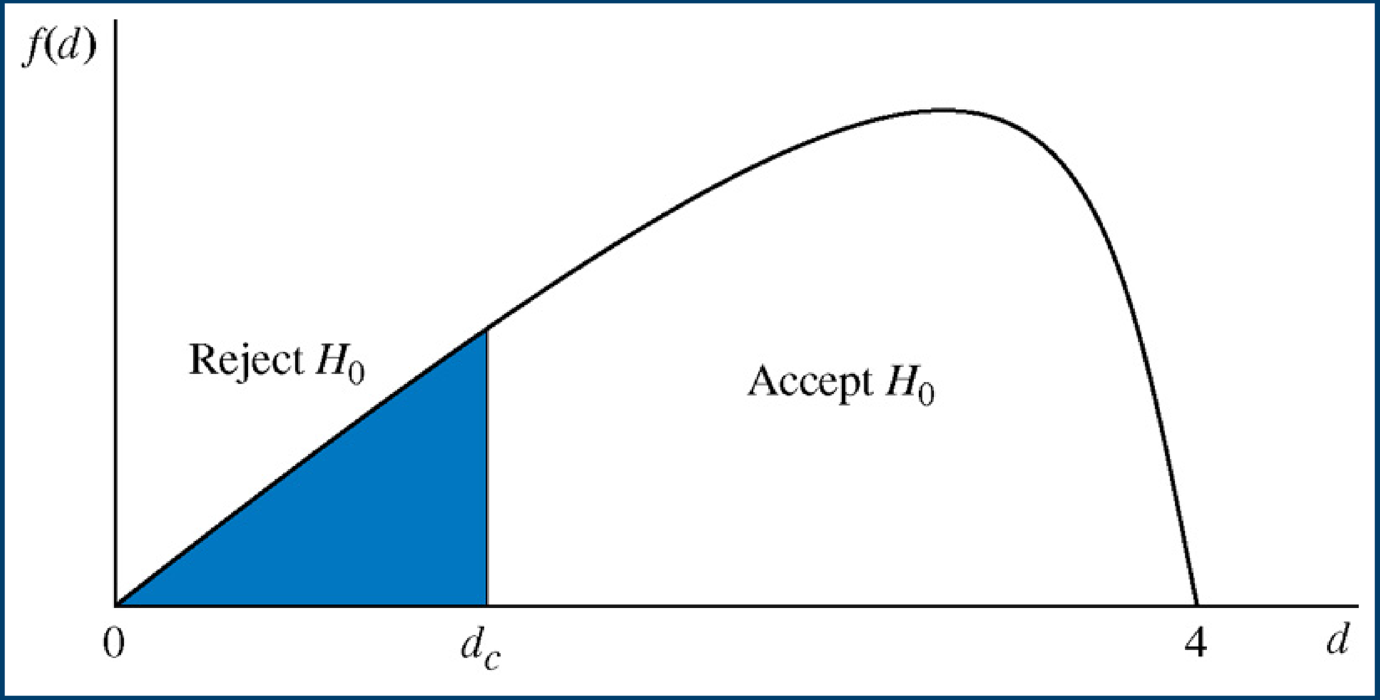

Figure 9A.1:

Appendix 9B
9B.1 The Durbin-Watson Bounds Test

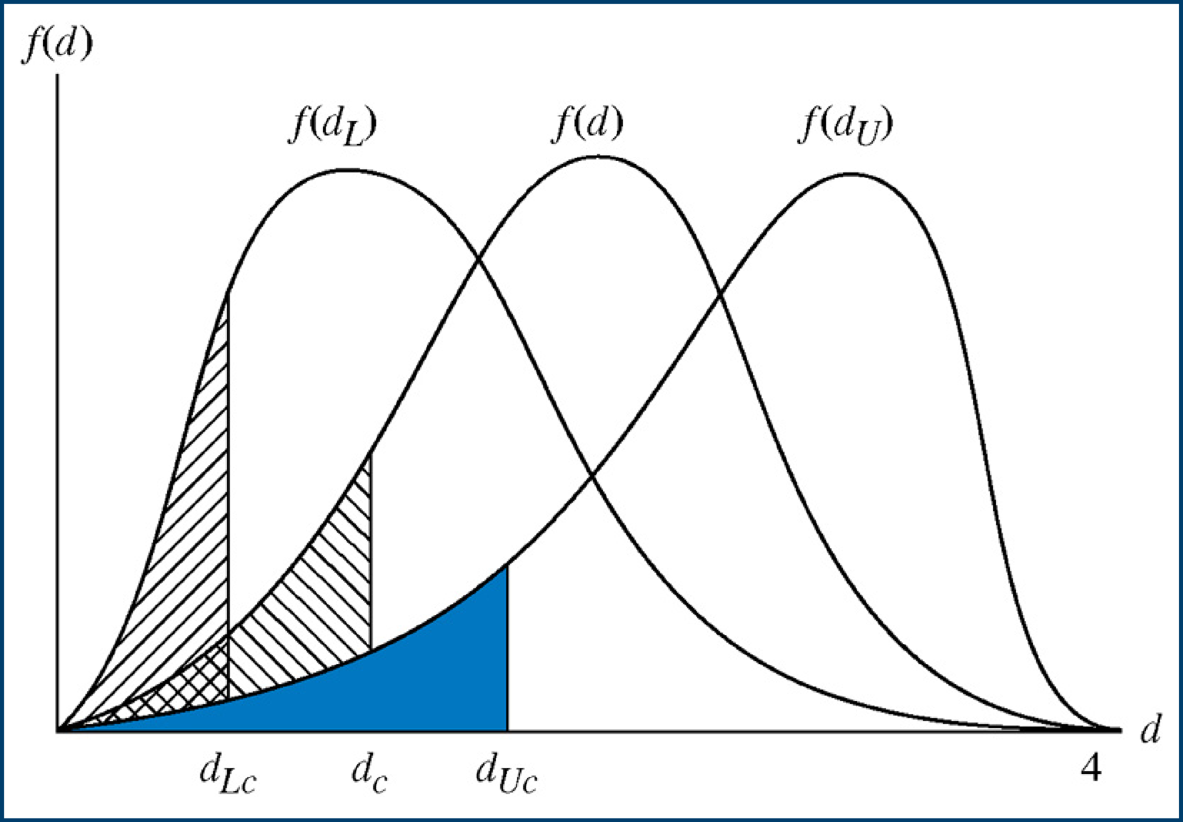

Figure 9A.2:

Appendix 9B
9B.1 The Durbin-Watson Bounds Test

The Durbin-Watson *bounds test*.

- if $d < d_{Lc}$, reject $H_0 : \rho = 0$ and accept $H_1 : \rho > 0$;
- if $d > d_{Uc}$, do not reject $H_0 : \rho = 0$;
- if $d_{Lc} < d < d_{Uc}$, the test is inconclusive.

Appendix 9C
Deriving ARDL Lag Weights

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \beta_3 x_{t-3} + \cdots + e_t = \alpha + \sum_{s=0}^{\infty} \beta_s x_{t-s} + e_t$$

$$y_t = \delta + \delta_0 x_t + \delta_1 x_{t-1} + \cdots + \delta_q x_{t-q} + \theta_1 y_{t-1} + \cdots + \theta_p y_{t-p} + v_t$$

Appendix 9C

9C.1 The Geometric Lag

$$y_t = \delta + \delta_0 x_t + \theta_1 y_{t-1} + v_t \qquad (9C.1)$$

$$y_{t-1} = \delta + \delta_0 x_{t-1} + \theta_1 y_{t-2} \qquad (9C.2)$$

$$y_t = \delta + \delta_0 x_t + \theta_1 y_{t-1} = \delta + \delta_0 x_t + \theta_1 (\delta + \delta_0 x_{t-1} + \theta_1 y_{t-2})$$

$$= \delta + \theta_1 \delta + \delta_0 x_t + \theta_1 \delta_0 x_{t-1} + \theta_1^2 y_{t-2}$$

Appendix 9C
9C.1 The Geometric Lag

$$y_t = \delta + \theta_1\delta + \delta_0 x_t + \theta_1\delta_0 x_{t-1} + \theta_1^2(\delta + \delta_0 x_{t-2} + \theta_1 y_{t-3})$$

$$= \delta + \theta_1\delta + \theta_1^2\delta + \delta_0 x_t + \theta_1\delta_0 x_{t-1} + \theta_1^2\delta_0 x_{t-2} + \theta_1^3 y_{t-3}$$

$$y_t = \delta + \theta_1\delta + \theta_1^2\delta + \cdots + \theta_1^j\delta$$

$$+ \delta_0 x_t + \theta_1\delta_0 x_{t-1} + \theta_1^2\delta_0 x_{t-2} + \cdots + \theta_1^j\delta_0 x_{t-j} + \theta_1^{j+1} y_{t-(j+1)} \qquad \text{(9C.3)}$$

$$= \delta(1 + \theta_1 + \theta_1^2 + \cdots + \theta_1^j) + \sum_{s=0}^{j}\delta_0\theta_1^s x_{t-s} + \theta_1^{j+1} y_{t-(j+1)}$$

Appendix 9C
9C.1 The Geometric Lag

$$y_t = \alpha + \sum_{s=0}^{\infty} \delta_0 \theta_1^s x_{t-s} \qquad (9C.4)$$

$$\alpha = \delta(1 + \theta_1 + \theta_1^2 + \cdots) = \frac{\delta}{1 - \theta_1}$$

$$y_t = \alpha + \sum_{s=0}^{\infty} \beta_s x_{t-s} + e_t$$

Appendix 9C
9C.1 The Geometric Lag

$$\beta_s = \delta_0 \theta_1^s$$

$$\sum_{s=0}^{\infty} \beta_s = \delta_0 (1 + \theta_1 + \theta_1^2 + \cdots) = \frac{\delta_0}{1 - \theta_1}$$

Appendix 9C
9C.2 Lag Weights for More General ARDL Models

$$y_t = \delta + \delta_0 x_t + \delta_1 x_{t-1} + \delta_2 x_{t-2} + \delta_3 x_{t-3} + \theta_1 y_{t-1} + \theta_2 y_{t-2} + v_t \quad (9C.5)$$

$$\begin{aligned}
\beta_0 &= \delta_0 \\
\beta_1 &= \theta_1\beta_0 + \delta_1 \\
\beta_2 &= \theta_1\beta_1 + \theta_2\beta_0 + \delta_2 \\
\beta_3 &= \theta_1\beta_2 + \theta_2\beta_1 + \delta_3 \\
\beta_4 &= \theta_1\beta_3 + \theta_2\beta_2 \\
&\vdots \\
\beta_s &= \theta_1\beta_{s-1} + \theta_2\beta_{s-2} \quad \text{for } s \geq 4
\end{aligned} \quad (9C.6)$$

Appendix 9D
Forecasting: Exponential Smoothing

$$\hat{y}_{T+1} = \frac{y_T + y_{T-1} + y_{T-2}}{3}$$

$$\hat{y}_{T+1} = \alpha y_T + \alpha(1-\alpha)^1 y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \cdots \qquad (9D.1)$$

$$(1-\alpha)\hat{y}_T = \alpha(1-\alpha)y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \alpha(1-\alpha)^3 y_{T-3} + \ldots.. \quad (9D.2)$$

$$\hat{y}_{T+1} = \alpha y_T + (1-\alpha) y_T$$

Appendix 9D
Forecasting: Exponential Smoothing

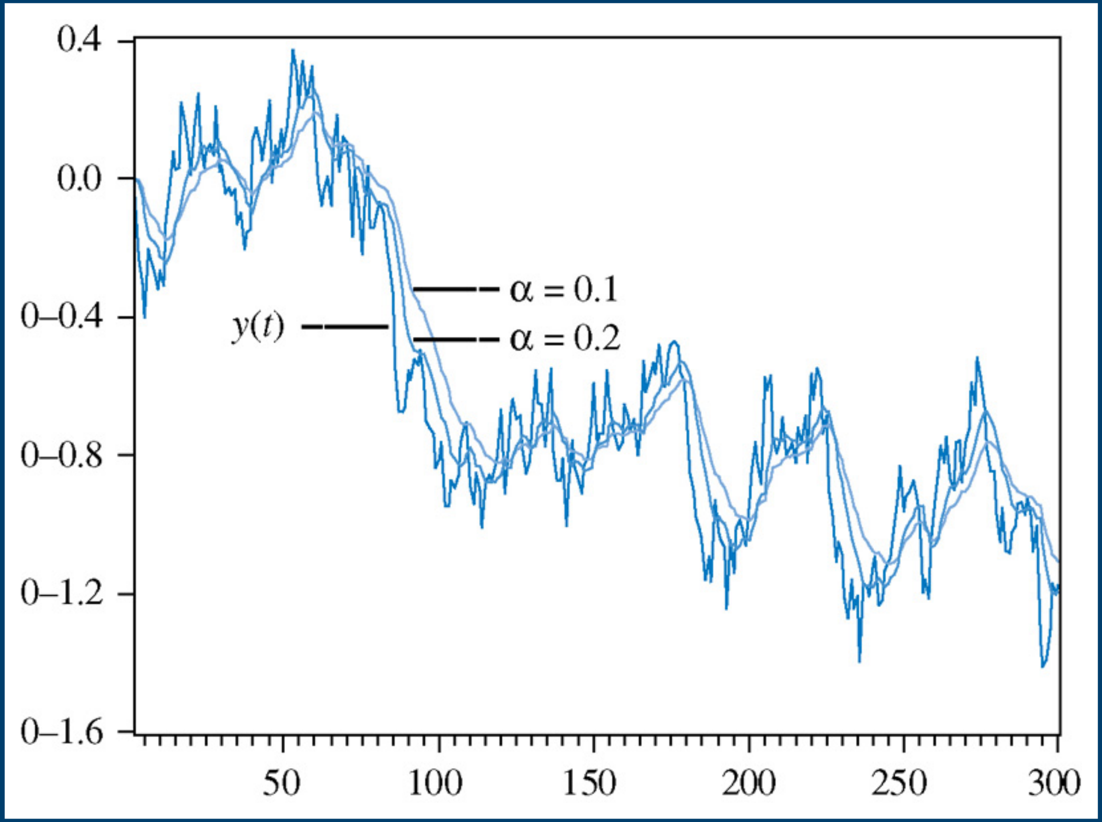

Figure 9A.3: Exponential Smoothing Forecasts for two alternative values of α