Interval Estimation and Hypothesis Testing

Chapter 3

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Chapter 3:
Interval Estimation and Hypothesis Testing

- 3.1 Interval Estimation
- 3.2 Hypothesis Tests
- 3.3 Rejection Regions for Specific Alternatives
- 3.4 Examples of Hypothesis Tests
- 3.5 The *p*-value

3.1 Interval Estimation

Assumptions of the Simple Linear Regression Model

- SR1. $y = \beta_1 + \beta_2 x + e$
- SR2. $E(e) = 0 \Leftrightarrow E(y) = \beta_1 + \beta_2 x$
- SR3. $\text{var}(e) = \sigma^2 = \text{var}(y)$
- SR4. $\text{cov}(e_i, e_j) = \text{cov}(y_i, y_j) = 0$
- SR5. The variable x is not random, and must take at least two different values.
- SR6. (*optional*) The values of e are *normally distributed* about their mean $e \sim N(0, \sigma^2)$

3.1.1 The t-distribution

- The normal distribution of b_2, the least squares estimator of β, is

$$b_2 \sim N\left(\beta_2, \frac{\sigma^2}{\sum(x_i - \bar{x})^2}\right)$$

- A standardized normal random variable is obtained from b_2 by subtracting its mean and dividing by its standard deviation:

$$Z = \frac{b_2 - \beta_2}{\sqrt{\sigma^2 / \sum(x_i - \bar{x})^2}} \sim N(0,1) \quad (3.1)$$

- The standardized random variable Z is normally distributed with mean 0 and variance 1.

3.1.1 The t-distribution

$$P\left(-1.96 \le Z \le 1.96\right) = .95$$

$$P\left(-1.96 \le \frac{b_2 - \beta_2}{\sqrt{\sigma^2 / \sum (x_i - \bar{x})^2}} \le 1.96\right) = .95$$

$$P\left(b_2 - 1.96\sqrt{\sigma^2 / \sum (x_i - \bar{x})^2} \le \beta_2 \le b_2 + 1.96\sqrt{\sigma^2 / \sum (x_i - \bar{x})^2}\right) = .95$$

This defines an interval that has probability .95 of containing the parameter β_2 .

3.1.1 The t-distribution

- The two endpoints $\left(b_2 \pm 1.96\sqrt{\sigma^2 / \sum(x_i - \bar{x})^2}\right)$ provide an interval estimator.

- In repeated sampling 95% of the intervals constructed this way will contain the true value of the parameter β_2.

- This easy derivation of an interval estimator is based on the assumption SR6 *and* that we know the variance of the error term σ^2.

3.1.1 The t-distribution

- Replacing σ^2 with $\hat{\sigma}^2$ creates a random variable *t:*

$$t = \frac{b_2 - \beta_2}{\sqrt{\hat{\sigma}^2 / \sum (x_i - \bar{x})^2}} = \frac{b_2 - \beta_2}{\sqrt{\widehat{\text{var}}(b_2)}} = \frac{b_2 - \beta_2}{\text{se}(b_2)} \sim t_{(N-2)} \quad (3.2)$$

- The ratio $t = (b_2 - \beta_2)/\text{se}(b_2)$ has a *t*-distribution with $(N - 2)$ degrees of freedom, which we denote as $t \sim t_{(N-2)}$.

3.1.1 The t-distribution

- In general we can say, if assumptions SR1-SR6 hold in the simple linear regression model, then

$$t = \frac{b_k - \beta_k}{\text{se}(b_k)} \sim t_{(N-2)} \text{ for } k = 1,2 \tag{3.3}$$

- The t-distribution is a bell shaped curve centered at zero.
- It looks like the standard normal distribution, except it is more spread out, with a larger variance and thicker tails.
- The shape of the t-distribution is controlled by a single parameter called the **degrees of freedom**, often abbreviated as *df*.

3.1.2 Obtaining Interval Estimates

- We can find a “critical value” from a t-distribution such that

$$P\left(t \geq t_c\right) = P\left(t \leq -t_c\right) = \alpha/2$$

where α is a probability often taken to be α = .01 or α = .05.

- The critical value t_c for degrees of freedom m is the percentile value $t_{(1-\alpha/2,m)}$.

3.1.2 Obtaining Interval Estimates

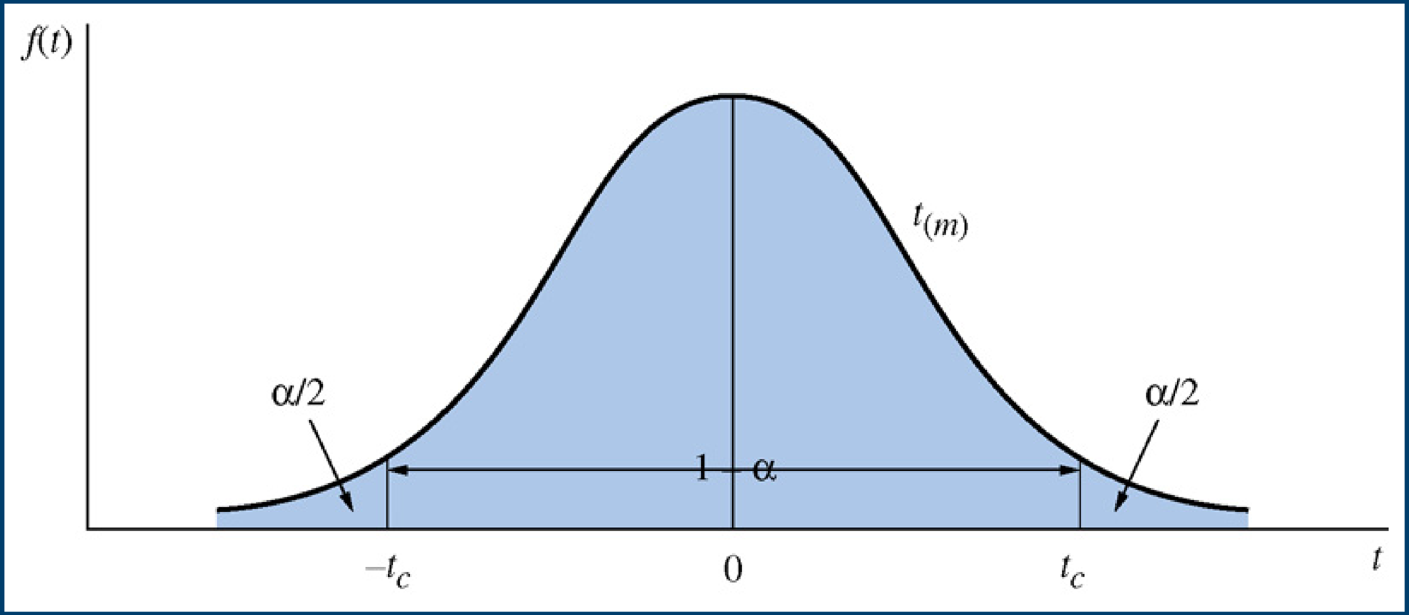

Figure 3.1 Critical Values from a *t*-distribution

3.1.2 Obtaining Interval Estimates

- Each shaded "tail" area contains $\alpha/2$ of the probability, so that $1-\alpha$ of the probability is contained in the center portion.
- Consequently, we can make the probability statement

$$P(-t_c \leq t \leq t_c) = 1 - \alpha \tag{3.4}$$

$$P[-t_c \leq \frac{b_k - \beta_k}{\text{se}(b_k)} \leq t_c] = 1 - \alpha$$

$$P[b_k - t_c \text{se}(b_k) \leq \beta_k \leq b_k + t_c \text{se}(b_k)] = 1 - \alpha \tag{3.5}$$

3.1.3 An Illustration

- For the food expenditure data

$$P[b_2 - 2.024\text{se}(b_2) \le \beta_2 \le b_2 + 2.024\text{se}(b_2)] = .95 \quad (3.6)$$

- The critical value $t_c = 2.024$, which is appropriate for $\alpha = .05$ and 38 degrees of freedom.
- To construct an interval estimate for β_2 we use the least squares estimate $b_2 = 10.21$ and its standard error

$$\text{se}(b_2) = \sqrt{\widehat{\text{var}}(b_2)} = \sqrt{4.38} = 2.09$$

3.1.3 An Illustration

- A "95% confidence interval estimate" for β_2:

$$b_2 \pm t_c \text{se}(b_2) = 10.21 \pm 2.024(2.09) = [5.97, 14.45]$$

When the procedure we used is applied to many random samples of data from the same population, then 95% of all the interval estimates constructed using this procedure will contain the true parameter.

3.1.4 The Repeated Sampling Context

Table 3.1 **Least Squares Estimates from 10 Random Samples**

Sample	b_1	se(b_1)	b_2	se(b_2)	$\hat{\sigma}^2$
1	131.69	40.58	6.48	1.96	7002.85
2	57.25	33.13	10.88	1.60	4668.63
3	103.91	37.22	8.14	1.79	5891.75
4	46.50	33.33	11.90	1.61	4722.58
5	84.23	41.15	9.29	1.98	7200.16
6	26.63	45.78	13.55	2.21	8911.43
7	64.21	32.03	10.93	1.54	4362.12
8	79.66	29.87	9.76	1.44	3793.83
9	97.30	29.14	8.05	1.41	3610.20
10	95.96	37.18	7.77	1.79	5878.71

3.1.4 The Repeated Sampling Context

Table 3.2 Interval Estimates from 10 Random Samples

Sample	$b_1 - t_c\text{se}(b_1)$	$b_1 + t_c\text{se}(b_1)$	$b_2 - t_c\text{se}(b_2)$	$b_2 + t_c\text{se}(b_2)$
1	49.54	213.85	2.52	10.44
2	−9.83	124.32	7.65	14.12
3	28.56	179.26	4.51	11.77
4	−20.96	113.97	8.65	15.15
5	0.93	167.53	5.27	13.30
6	−66.04	119.30	9.08	18.02
7	−0.63	129.05	7.81	14.06
8	19.19	140.13	6.85	12.68
9	38.32	156.29	5.21	10.89
10	20.69	171.23	4.14	11.40

3.2 Hypothesis Tests

Components of Hypothesis Tests

1. A null hypothesis, H_0
2. An alternative hypothesis, H_1
3. A test statistic
4. A rejection region
5. A conclusion

3.2 Hypothesis Tests

- The Null Hypothesis

The null hypothesis, which is denoted H_0 (*H-naught*), specifies a value for a regression parameter.

The null hypothesis is stated $H_0 : \beta_k = c$, where *c* is a constant, and is an important value in the context of a specific regression model.

3.2 Hypothesis Tests

- The Alternative Hypothesis

Paired with every null hypothesis is a logical alternative hypothesis, H_1, that we will accept if the null hypothesis is rejected.

For the null hypothesis H_0: $\beta_k = c$ the three possible alternative hypotheses are:

- $H_1 : \beta_k > c$
- $H_1 : \beta_k < c$
- $H_1 : \beta_k \neq c$

3.2 Hypothesis Tests

- The Test Statistic

$$t = \left(b_k - \beta_k\right) / \mathrm{se}(b_k) \sim t_{(N-2)}$$

- ***If*** the null hypothesis $H_0 : \beta_k = c$ is *true*, ***then*** we can substitute c for β_k and it follows that

$$t = \frac{b_k - c}{\mathrm{se}(b_k)} \sim t_{(N-2)} \qquad (3.7)$$

If the null hypothesis is *not true*, then the *t*-statistic in (3.7) does *not* have a *t*-distribution with $N - 2$ degrees of freedom.

3.2 Hypothesis Tests

- The Rejection Region

The rejection region depends on the form of the alternative. It is the range of values of the test statistic that leads to *rejection* of the null hypothesis. It is possible to construct a rejection region only if we have:

- a test statistic whose distribution is known when the null hypothesis is true
- an alternative hypothesis
- a level of significance

The level of significance α is usually chosen to be .01, .05 or .10.

3.2 Hypothesis Tests

- A Conclusion

We make a correct decision if:

- The null hypothesis is *false* and we decide to *reject* it.
- The null hypothesis is *true* and we decide *not* to reject it.

Our decision is incorrect if:

- The null hypothesis is *true* and we decide to *reject* it (a Type I error)
- The null hypothesis is *false* and we decide *not* to reject it (a Type II error)

3.3 Rejection Regions for Specific Alternatives

- 3.3.1. One-tail Tests with Alternative “Greater Than” ($>$)
- 3.3.2. One-tail Tests with Alternative “Less Than” ($<$)
- 3.3.3. Two-tail Tests with Alternative “Not Equal To” ($\neq$)

3.3.1 One-tail Tests with Alternative "Greater Than" (>)

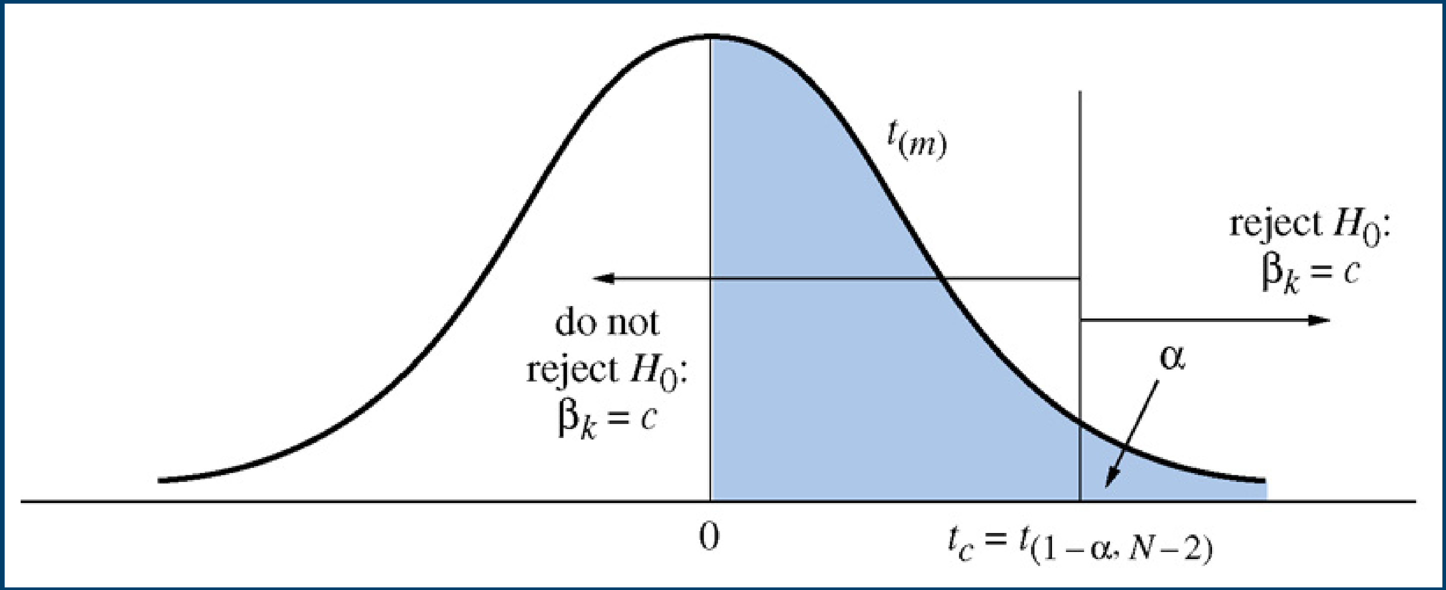

Figure **3.2** Rejection region for a one-tail test of H_0: $\beta_k = c$ against H_1: $\beta_k > c$

3.3.1 One-tail Tests with Alternative "Greater Than" (>)

When testing the null hypothesis $H_0 : \beta_k = c$ against the alternative hypothesis $H_1 : \beta_k > c$, reject the null hypothesis and accept the alternative hypothesis if $t \geq t_{(1-\alpha, N-2)}$.

3.3.2 One-tail Tests with Alternative "Less Than" (<)

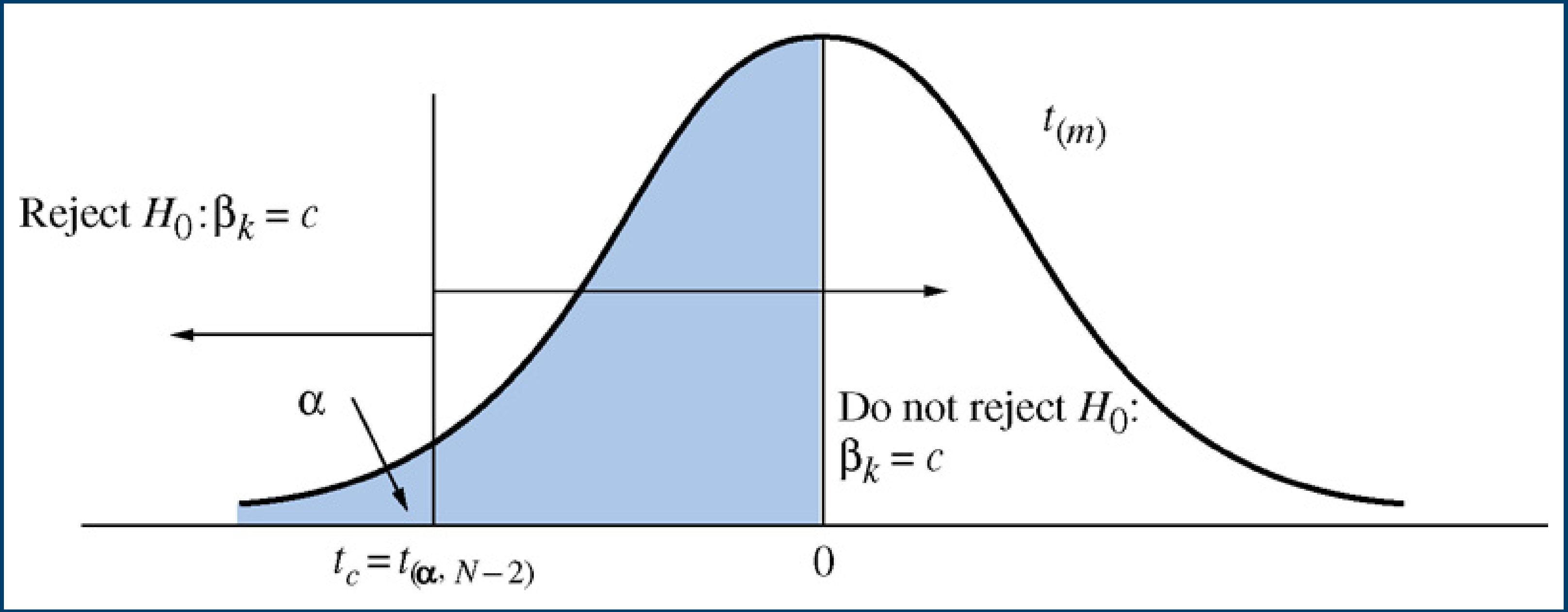

Figure **3.3** The rejection region for a one-tail test of H_0: $\beta_k = c$ against H_1: $\beta_k < c$

3.3.2 One-tail Tests with Alternative "Less Than" (<)

When testing the null hypothesis $H_0 : \beta_k = c$ against the alternative hypothesis $H_1 : \beta_k < c$, reject the null hypothesis and accept the alternative hypothesis if $t \le t_{(\alpha, N-2)}$.

3.3.3 Two-tail Tests with Alternative "Not Equal To" (≠)

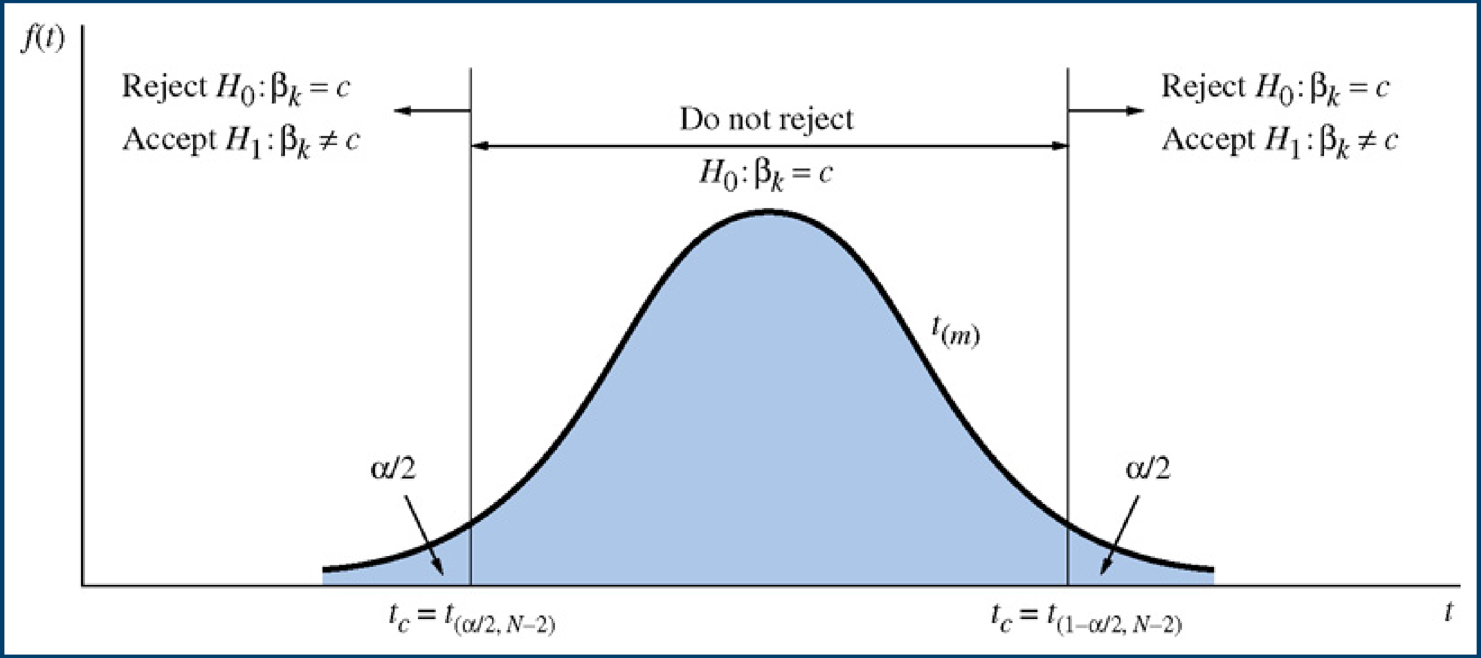

Figure **3.4** The rejection region for a two-tail test of H_0: $\beta_k = c$ against H_1: $\beta_k \neq c$

3.3.3 Two-tail Tests with Alternative "Not Equal To" (≠)

When testing the null hypothesis $H_0 : \beta_k = c$ against the alternative hypothesis $H_1 : \beta_k \neq c$, reject the null hypothesis and accept the alternative hypothesis if $t \leq t_{(\alpha/2, N-2)}$ **or** if $t \geq t_{(1-\alpha/2, N-2)}$.

3.4 Examples of Hypothesis Tests

STEP-BY-STEP PROCEDURE FOR TESTING HYPOTHESES

1. Determine the null and alternative hypotheses.
2. Specify the test statistic and its distribution if the null hypothesis is true.
3. Select α and determine the rejection region.
4. Calculate the sample value of the test statistic.
5. State your conclusion.

3.4.1 Right-tail Tests

3.4.1a One-tail Test of Significance

1. The null hypothesis is $H_0 : \beta_2 = 0$.

   The alternative hypothesis is $H_1 : \beta_2 > 0$.

2. The test statistic is (3.7). In this case $c = 0$, so $t = b_2 / \text{se}(b_2) \sim t_{(N-2)}$

   if the null hypothesis is true.

3. Let us select $\alpha = .05$. The critical value for the right-tail rejection region is the 95th percentile of the t-distribution with $N - 2 = 38$ degrees of freedom, $t_{(95,38)}) = 1.686$. Thus we will reject the null hypothesis if the calculated value of $t \geq 1.686$. If $t < 1.686$, we will not reject the null hypothesis.

3.4.1 Right-tail Tests

4. Using the food expenditure data, we found that $b_2 = 10.21$ with standard error $\text{se}(b_2) = 2.09$. The value of the test statistic is

$$t = \frac{b_2}{\text{se}(b_2)} = \frac{10.21}{2.09} = 4.88$$

5. Since $t = 4.88 > 1.686$, we reject the null hypothesis that $\beta_2 = 0$ and accept the alternative that $\beta_2 > 0$. That is, we reject the hypothesis that there is no relationship between income and food expenditure, and conclude that there is a *statistically significant* positive relationship between household income and food expenditure.

3.4.1 Right-tail Tests

- 3.4.1b One-tail Test of an Economic Hypothesis

1. The null hypothesis is $H_0 : \beta_2 \le 5.5$.

   The alternative hypothesis is $H_1 : \beta_2 > 5.5$.

2. The test statistic $t = \left(b_2 - 5.5\right) / \mathrm{se}\left(b_2\right) \sim t_{(N-2)}$

   if the null hypothesis is true.

3. Let us select $\alpha = .01$. The critical value for the right-tail rejection region is the 99th percentile of the *t*-distribution with $N - 2 = 38$ degrees of freedom, $t_{(99,38)} = 2.429$. We will reject the null hypothesis if the calculated value of $t \ge 2.429$. If $t < 2.429$, we will not reject the null hypothesis.

3.4.1 Right-tail Tests

4. Using the food expenditure data, $b_2 = 10.21$ with standard error se(b_2) = 2.09. The value of the test statistic is

$$t = \frac{b_2 - 5.5}{\text{se}(b_2)} = \frac{10.21 - 5.5}{2.09} = 2.25$$

5. Since $t = 2.25 < 2.429$ we do not reject the null hypothesis that $\beta_2 \le 5.5$. We are *not* able to conclude that the new supermarket will be profitable and will not begin construction.

3.4.2 Left-tail Tests

1. The null hypothesis is $H_0 : \beta_2 \geq 15$.

   The alternative hypothesis is $H_1 : \beta_2 < 15$.

2. The test statistic $t = (b_2 - 15)/\text{se}(b_2) \sim t_{(N-2)}$

   if the null hypothesis is true.

3. Let us select α = .05. The critical value for the left-tail rejection region is the 5th percentile of the *t*-distribution with $N - 2 = 38$ degrees of freedom, $t_{(05,38)} = -1.686$. We will reject the null hypothesis if the calculated value of $t \leq -1.686$. If $t > -1.686$, we will not reject the null hypothesis.

3.4.2 Left-tail Tests

4. Using the food expenditure data, $b_2 = 10.21$ with standard error se$(b_2) = 2.09$. The value of the test statistic is

$$t = \frac{b_2 - 15}{\text{se}(b_2)} = \frac{10.21 - 15}{2.09} = -2.29$$

5. Since $t = -2.29 < -1.686$ we reject the null hypothesis that $\beta_2 \geq 15$ and accept the alternative that $\beta_2 < 15$. We conclude that households spend less than \$15 from each additional \$100 income on food.

3.4.3 Two-tail Tests

- 3.4.3a Two-tail Test of an Economic Hypothesis

1. The null hypothesis is $H_0 : \beta_2 = 7.5$.

   The alternative hypothesis is $H_1 : \beta_2 \neq 7.5$.

2. The test statistic $t = (b_2 - 7.5)/\text{se}(b_2) \sim t_{(N-2)}$

   if the null hypothesis is true.

3. Let us select α = .05. The critical values for this two-tail test are the 2.5-percentile $t_{(.025,38)} = -2.024$ and the 97.5-percentile $t_{(.975,38)} = 2.024$. Thus we will reject the null hypothesis if the calculated value of $t \geq 2.024$ **or** if $t \leq -2.024$. If $-2.024 < t < 2.024$, we will not reject the null hypothesis.

3.4.3 Two-tail Tests

4. Using the food expenditure data, $b_2 = 10.21$ with standard error se(b_2) = 2.09. The value of the test statistic is

$$t = \frac{b_2 - 7.5}{\text{se}(b_2)} = \frac{10.21 - 7.5}{2.09} = 1.29$$

5. Since $-2.204 < t = 1.29 < 2.204$ we do not reject the null hypothesis that $\beta_2 = 7.5$. The sample data are consistent with the conjecture households will spend an additional \$7.50 per additional \$100 income on food.

3.4.3 Two-tail Tests

3.4.3b Two-tail Test of Significance

1. The null hypothesis is $H_0 : \beta_2 = 0$.

   The alternative hypothesis is $H_1 : \beta_2 \neq 0$.

2. The test statistic $t = b_2 / \text{se}(b_2) \sim t_{(N-2)}$

   if the null hypothesis is true.

3. Let us select α = .05. The critical values for this two-tail test are the 2.5-percentile $t_{(.025,38)} = -2.024$ and the 97.5-percentile $t_{(.975,38)} = 2.024$. Thus we will reject the null hypothesis if the calculated value of $t \geq 2.024$ **or** if $t \leq -2.024$. If $-2.024 < t < 2.024$, we will not reject the null hypothesis.

3.4.3 Two-tail Tests

4. Using the food expenditure data, $b_2 = 10.21$ with standard error $\text{se}(b_2) = 2.09$. The value of the test statistic is

$$t = \frac{b_2}{\text{se}(b_2)} = \frac{10.21}{2.09} = 4.88$$

5. Since $t = 4.88 > 2.204$ we reject the null hypothesis that $\beta_2 = 0$ and conclude that there is a statistically significant relationship between income and food expenditure.

3.4.3 Two-tail Tests

Coefficient	Variable	Std. Error	t-Statistic	Prob.
83.41600	*C*	43.41016	1.921578	0.0622
10.20964	*INCOME*	2.093264	4.877381	0.0000

3.5 The *p*-Value

***p*-value rule**: Reject the null hypothesis when the *p*-value is less than, or equal to, the level of significance α. That is, if $p \leq \alpha$ then reject H_0. If $p > \alpha$ then do not reject H_0.

3.5 The *p*-Value

If t is the calculated value of the t-statistic, then:

- if H_1: $\beta_K > c$, p = probability to the right of t
- if H_1: $\beta_K < c$, p = probability to the left of t
- if H_1: $\beta_K \neq c$, p = sum of probabilities to the right of $|t|$ and to the left of $-|t|$

3.5.1 *p*-value for a Right-tail Test

Recall section 3.4.1b:

- The null hypothesis is $H_0: \beta_2 \le 5.5$.

  The alternative hypothesis is $H_1: \beta_2 > 5.5$.

$$t = \frac{b_2 - 5.5}{\text{se}(b_2)} = \frac{10.21 - 5.5}{2.09} = 2.25$$

- If $F_X(x)$ is the *cdf* for a random variable X, then for any value $x=c$ the cumulative probability is $P[X \le c] = F_X(c)$.
- $p = P[t_{(38)} \ge 2.25] = 1 - P[t_{(38)} \le 2.25] = 1 - .9848 = .0152$

3.5.1 *p*-value for a Right-tail Test

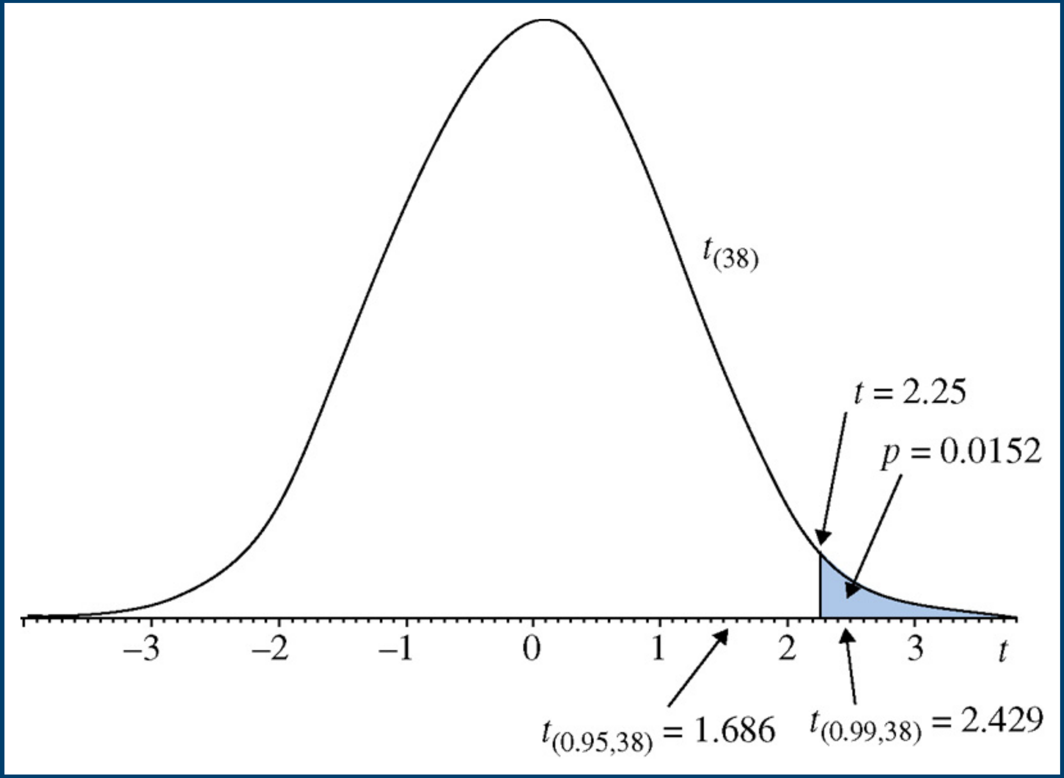

Figure **3.5** The *p*-value for a right tail test

3.5.2 *p*-value for a Left-tail Test

Recall section 3.4.2:

- The null hypothesis is $H_0: \beta_2 \geq 15$.

  The alternative hypothesis is $H_1: \beta_2 < 15$.

$$t = \frac{b_2 - 15}{\text{se}(b_2)} = \frac{10.21 - 15}{2.09} = -2.29$$

- $$P\left[t_{(38)} \leq -2.29\right] = .0139$$

3.5.2 *p*-value for a Left-tail Test

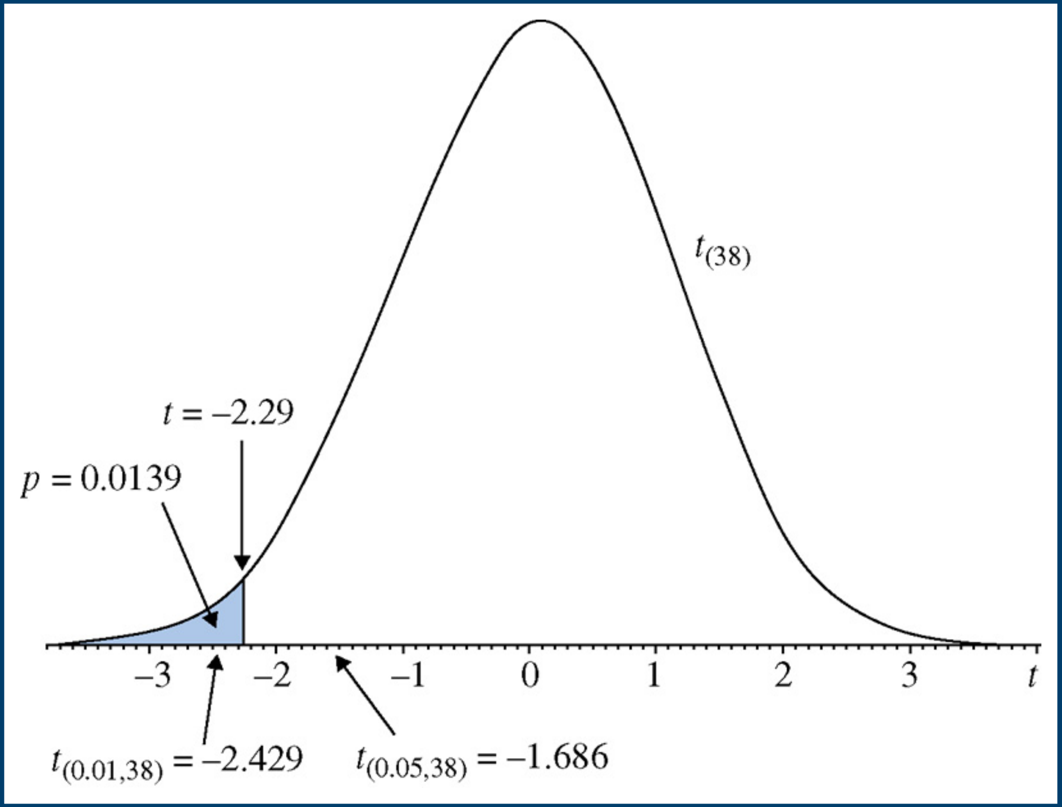

Figure **3.6** The *p*-value for a left tail test

3.5.3 *p*-value for a Two-tail Test

Recall section 3.4.3a:

- The null hypothesis is H_0: $\beta_2 = 7.5$.

  The alternative hypothesis is H_1: $\beta_2 \neq 7.5$.

  $$t = \frac{b_2 - 7.5}{\text{se}(b_2)} = \frac{10.21 - 7.5}{2.09} = 1.29$$

- $$p = P\left[t_{(38)} \geq 1.29\right] + P\left[t_{(38)} \leq -1.29\right] = .2033$$

3.5.3 *p*-value for a Two-tail Test

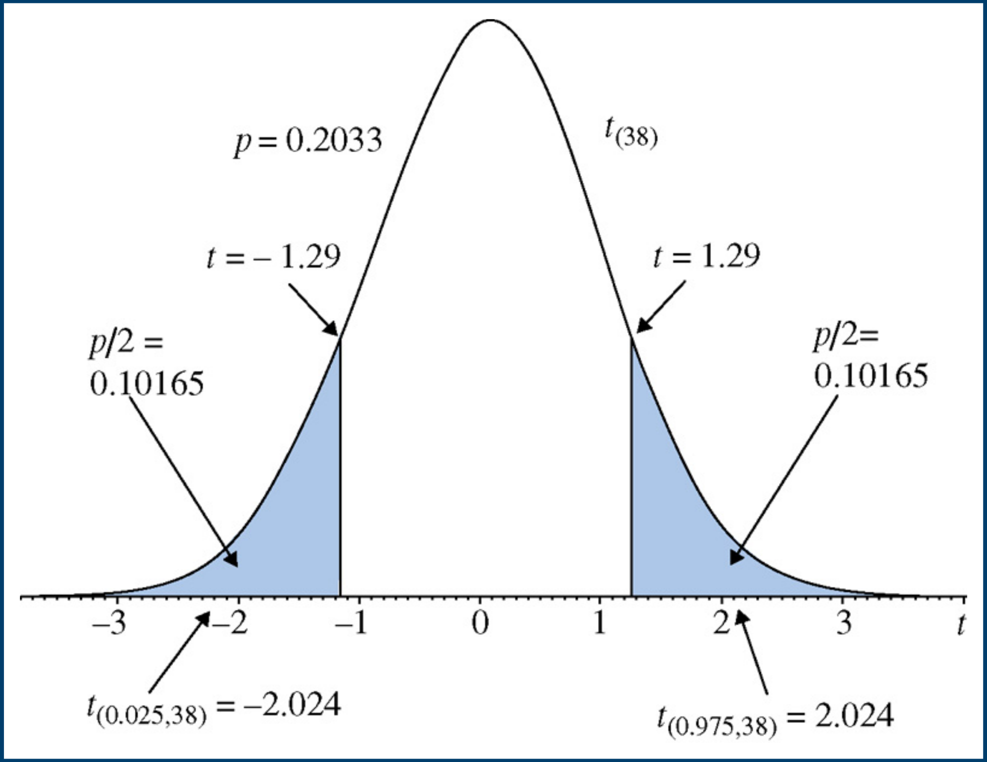

Figure **3.7** The *p*-value for a two-tail test

3.5.4 *p*-value for a Two-tail Test of Significance

Recall section 3.4.3b:

- The null hypothesis is H_0: $\beta_2 = 0$.

  The alternative hypothesis is H_1: $\beta_2 \neq 0$

- $$p = P\left[t_{(38)} \geq 4.88\right] + P\left[t_{(38)} \leq -4.88\right] = 0.0000$$

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	83.41600	43.41016	1.921578	0.0622
INCOME	10.20964	2.093264	4.877381	0.0000

Keywords

- alternative hypothesis
- confidence intervals
- critical value
- degrees of freedom
- hypotheses
- hypothesis testing
- inference
- interval estimation
- level of significance
- null hypothesis
- one-tail tests
- point estimates
- probability value
- *p*-value
- rejection region
- test of significance
- test statistic
- two-tail tests
- Type I error
- Type II error

Chapter 3 Appendices

- Appendix 3A Derivation of the *t*-distribution
- Appendix 3B Distribution of the *t*-statistic under H_1

Appendix 3A
Derivation of the *t*-distribution

$$b_2 \sim N\left(\beta_2, \frac{\sigma^2}{\sum (x_i - \bar{x})^2}\right)$$

$$Z = \frac{b_2 - \beta_2}{\sqrt{\text{var}(b_2)}} \sim N(0,1) \qquad (3A.1)$$

$$\sum \left(\frac{e_i}{\sigma}\right)^2 = \left(\frac{e_1}{\sigma}\right)^2 + \left(\frac{e_2}{\sigma}\right)^2 + \cdots + \left(\frac{e_N}{\sigma}\right)^2 \sim \chi^2_{(N)} \qquad (3A.2)$$

$$V = \frac{\sum \hat{e}_i^2}{\sigma^2} = \frac{(N-2)\hat{\sigma}^2}{\sigma^2} \qquad (3A.3)$$

Appendix 3A
Derivation of the *t*-distribution

$$V = \frac{(N-2)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{(N-2)} \qquad (3A.4)$$

$$t = \frac{Z}{\sqrt{V/(N-2)}} = \frac{(b_2 - \beta_2)\Big/\sqrt{\sigma^2 \Big/ \sum (x_i - \bar{x})^2}}{\sqrt{\dfrac{(N-2)\hat{\sigma}^2/\sigma^2}{N-2}}}$$

$$= \frac{b_2 - \beta_2}{\sqrt{\dfrac{\hat{\sigma}^2}{\sum (x_i - \bar{x})^2}}} = \frac{b_2 - \beta_2}{\sqrt{\widehat{\operatorname{var}}(b_2)}} = \frac{b_2 - \beta_2}{\operatorname{se}(b_2)} \sim t_{(N-2)} \qquad (3A.5)$$

Appendix 3B
Distribution of the t-statistic under H_1

$$t = \frac{b_2 - 1}{\text{se}(b_2)} \sim t_{(N-2)}$$

$$\frac{b_2 - c}{\sqrt{\text{var}(b_2)}} \sim N\left(\frac{1-c}{\sqrt{\text{var}(b_2)}}, 1\right) \tag{3B.1}$$

where $$\text{var}(b_2) = \frac{\sigma^2}{\sum (x_i - \bar{x})^2}$$