Optimal Portfolio Selection

We have geometrically described characteristics of the optimal portfolio. Now we turn our attention to a methodology for exactly identifying the optimal portfolio given a set (or universe) of risky assets. Before we solve the planning problem, we know what the solution will look like from our geometric analysis. People will hold the risk free assets and a portion of the market portfolio of risky assets. But this insight does not tell us much about the risk of individual securities -- it just tells us what is held in equilibrium.

First, let's consider some of the data and assumptions that we need to solve the portfolio selection problem. First, we need to collect expected returns on all assets $E$. Second, we need to collect the variances and covariances of all assets' returns $V$. With these data, we can construct the minimum variance frontier and the efficient frontier of all risky assets (positively sloped portion of the minimum variance frontier). Third, we need riskless borrowing and lending at the rate $R_f$. With this additional data, we can construct the capital market line. This line will describe all the efficient portfolios that are a combination of two funds: the riskless asset and the tangency portfolio. The final step has to do with individuals' beliefs about the means and standard deviations of the assets' returns. If there is complete agreement or homogeneous expectations, then all investors will agree on the means and standard deviations. This implies that they all see the same efficient frontier of risky assets and they will all see the same capital market line once the riskless asset is introduced.

A critical ingredient is that the expected returns, volatilities and covariances must be expected. When I say "collect" these variables, this is an involved process relating to forecasting. It is unlikely that historical averages will be appropriate forecasts of the future. This forecasting task is the topic of the Global Tactical Asset Allocation class.
The Solution to the Portfolio Selection Problem

The Appendix section to this lecture details the derivation of the planning problem. The results are stated in this section without proof. Before the results are stated, we have to review some definitions.

The problem is to minimize portfolio variance subject to two constraints. The first constraint sets the level of expected return. Remember part of our definition of an efficient portfolio stated that you minimized variance for a given level of expected return. The first constraint sets this "given" level of expected return. The second constraint makes sure that the sum of the portfolio weights, $w$, is one. When this problem is solved, the efficient frontier portfolio weights are:

$$
1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}
$$

The optimal combination of risky assets or the tangency portfolio can be found as:

$$
w = \lambda_1 \Sigma^{-1} \mu + \lambda_2 \Sigma^{-1} 1
$$

where $\lambda_1$ and $\lambda_2$ are Lagrangian multipliers. By varying these multipliers, the entire efficient frontier of risky assets is traced out.

The capital market line can also be derived. Define:

$$
w = \frac{1}{1' \Sigma^{-1} (\mu - R_f 1)} [\Sigma^{-1} (\mu - R_f 1)]
$$

Note that the sum of the portfolio weights $w' 1$ equals one.
With this definition, the capital market line is:

$$\mu = R_f + \sqrt{k} \sigma$$

That is, with various values of sigma (portfolio standard deviation), a straight line is traced in mean-standard deviation space originating at $R_f$. A proof is found in Ingersoll, *Theory of Financial Decision Making* (1987, p.89).

The quadratic programming problem is the following:

$$\text{Min}_{\{w\}} \ w'Vw$$

Subject to

$$w'\mu = \mu^*$$

and

$$w'1 = 1$$

The Lagrangian is:

$$\frac{w'Vw}{2} + \lambda_1 (\mu^* - w'\mu) + \lambda_2 (1 - w'1)$$

The $\lambda_1$ and $\lambda_2$ terms are Lagrangian multipliers. The first order necessary conditions result from differentiating with respect to the portfolio weights:

$$FOC: \ w'V - \lambda_1 \mu - \lambda_2 1$$

Solving for expected return:

$$\mu = \frac{\lambda_2}{\lambda_1} + \frac{w'V}{\lambda_1}$$

$$= \gamma_0 + \gamma_1 w'V$$
This means that the expected return for any security is linear in its covariance with all the other securities. We can also solve the first order conditions for the optimal weights:

\[ w = \lambda_1 V^{-1} \mu + \lambda_2 V^{-1} \mu \]

The next step would be to solve for the Lagrangian multipliers. This is done by considering the constraints as a set of two equations and solving for the unknown multipliers after we substitute the above expression for the optimal weights into the first order conditions.

The optimal combination of risky assets (the tangency portfolio) is

\[ w^m = \left( \frac{1}{1' V^{-1} (\mu - r_f 1)} \right) V^{-1} (\mu - r_f 1) \]

Note that the weights sum to one.

The capital market line can also be derived. Define

\[ \kappa = (\mu - r_f 1)' V^{-1} (\mu - r_f 1) \]

With this definition, the capital market line is:

\[ \mu = r_f + (\sqrt{\kappa}) \sigma \]

That is, with the various values of the portfolio standard deviation (sigma), a straight line is traced in mean-standard deviation space originating at \( r_f \).

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Much of the material for this lecture is drawn from Douglas Breeden, "Portfolio Statistics".