Chapters 4: mixed, correlated, and Bayesian equilibrium

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1   Nash's theorem

Nash's theorem generalizes Von Neumann's theorem to n-person games.

Theorem 1 *(Nash) If in the game $\mathcal{G} = (N, S_i, u_i, i \in N)$ the sets S_i are convex and compact, and the functions u_i are continuous over X and quasi-concave in s_i, then the game has at least one Nash equilibrium.*

For the proof we use the following **mathematical preliminaries.**

1) Upper hemi-continuity of correspondences
A correspondence $f : A \rightarrow\rightarrow \mathbb{R}^m$ is called *upper hemicontinuous* at $x \in A$ if for any open set U such that $f(x) \subset U \subset A$ there exists an open set V such that $x \in V \subset A$ and that for any $y \in V$ we have $f(y) \subset U$. A correspondence $f : A \rightarrow\rightarrow \mathbb{R}^m$ is called *upper hemicontinuous* if it is upper hemicontinuous at all $x \in A$.

Note that for a single-valued function f, this definition is just the continuity of f.

Proposition 2 *A correspondence $f : A \rightarrow\rightarrow \mathbb{R}^m$ is upper hemicontinuous if and only if it has a closed graph and the images of the compact sets are bounded (i.e. for any compact $B \subset A$ the set $f(B) = \{y \in \mathbb{R}^m : y \in f(x)$ for some $x \in B\}$ is bounded).*

Note that if $f(A)$ is bounded (compact), then the upper hemicontinuity is equivalent to the closed graph condition. Thus to check that $f : A \rightarrow\rightarrow A$ from the premises of Kakutani's fixed point theorem is upper hemicontinuous it is enough to check that it has closed graph. I.e., one needs to check that for any $x^k \in A$, $x^k \to x \in A$, and for any $y^k \to y$ such that $y^k \in f(x^k)$, we have $y \in f(x)$.

2) Two fixed point theorems

Theorem 3 *(Brouwer's fixed point theorem) Let $A \subset \mathbb{R}^n$ be a nonempty convex compact, and $f : A \to A$ be single-valued and continuous. Then f has a fixed point : there exists $x \in A$ such that $x = f(x)$.*

Extension to correspondences:

Theorem 4 *(Kakutani's fixed point theorem)*

Let $A \subset \mathbb{R}^n$ be a nonempty convex compact and $f : A \rightarrow\rightarrow A$ be an upper hemicontinuous convex-valued correspondence such that $f(x) \neq \varnothing$ for any $x \in A$. Then f has a fixed point: there exists $x \in A$ such that $x \in f(x)$.

Proof of Nash Theorem.

For each player $i \in N$ define a best reply correspondence $R_i : S_{-i} \rightarrow\rightarrow S_i$ in the following way: $R_i(s_{-i}) = \arg\max\limits_{\sigma \in S_i} u_i(\sigma, s_{-i})$. Consider next the best reply correspondence $R : S \rightarrow\rightarrow S$, where $R(s) = R_1(s_{-1}) \times ... \times R_N(s_{-N})$. We will check that R satisfies the premises of the Kakutani's fixed point theorem.

First $S = S_1 \times ... \times S_N$ is a nonempty convex compact as a Cartesian product of finite number of nonempty convex compact subsets of $\mathbb{R}^p$.

Second since u_i are continuous and S_i are compact there always exist $\max\limits_{\sigma \in S_i} u_i(\sigma, s_{-i})$. Thus $R_i(s_{-i})$ is nonempty for any $s_{-i} \in S_{-i}$ and so $R(s)$ is nonempty for any $s \in S$.

Third $R(s) = R_1(s_{-1}) \times ... \times R_N(s_{-N})$ is convex since $R_i(s_{-i})$ are convex. The last statement follows from the (quasi-) concavity of $u_i(\cdot, s_{-i})$. Indeed if $s_i, t_i \in R_i(s_{-i}) = \arg\max\limits_{\sigma \in S_i} u_i(\sigma, s_{-i})$ then $u_i(\lambda s_i + (1-\lambda)t_i, s_{-i}) \geq \lambda u_i(s_i, s_{-i}) + (1-\lambda)u_i(t_i, s_{-i}) = \max\limits_{\sigma \in S_i} u_i(\sigma, s_{-i})$, and hence $\lambda s_i + (1-\lambda)t_i \in R_i(s_{-i})$.

Finally given that S is compact to guarantee upper hemicontinuity of R we only need to check that it has closed graph. Let $s^k \in S$, $s^k \rightarrow s \in S$, and $t^k \rightarrow t$ be such that $t^k \in R(s^k)$. Hence for any k and for any $i = 1, ..., N$ we have that $u_i(t^k, s^k_{-i}) \geq u_i(\sigma, s^k_{-i})$ for all $\sigma \in S_i$. Given that $(t^k, s^k_{-i}) \rightarrow (t, s_{-i})$ continuity of u_i implies that $u_i(t, s_{-i}) \geq u_i(\sigma, s_{-i})$ for all $\sigma \in S_i$. Thus $t \in \arg\max\limits_{\sigma \in S_i} u_i(\sigma, s_{-i}) = R(s)$ and so R has closed graph.

Now, Kakutani's fixed point theorem tells us that there exists $s \in S = S_1 \times ... \times S_N$ such that $s = (s_1, ..., s_N) \in R(s) = R_1(s_{-1}) \times ... \times R_N(s_{-N})$. I.e. $s_i \in R(s_{-i})$ for all players i. Hence, each strategy in s is a best reply to the vector of strategies of other players and thus s is a Nash equilibrium of our game.■

A useful variant of the theorem is for symmetrical games.

Theorem 5 *If in addition to the above assumptions, the game is symmetrical, then there exists a symmetrical Nash equilibrium $s_i = s_j$ for all i, j.*

Proof. The game is (N, S_0, u) with S_0 the common strategy set, and $u : S_0 \times S_0^{N \setminus \{1\}} \rightarrow \mathbb{R}$ its common payoff function. Check that we can apply Kakutani's theorem to the mapping R_0 from S_0 into itself:

$$R_0(s_0) = \arg\max\limits_{\sigma \in S_0} u_i(\sigma; s_0, s_0, \cdots, s_0)$$

A fixed point of R_0 is a symmetric Nash equilibrium. ■

The main application of Nash's theorem is to finite games in strategic form where the players use mixed strategies.

Consider a normal form game $\Gamma_f = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, where N is a (finite) set of players, C_i is the (nonempty) **finite** set of **pure** strategies available to the player i, and $u_i : C = C_1 \times ... \times C_N \to \mathbb{R}$ is the payoff function for player i. Let $S_i = \Delta(C_i)$ be the set of all probability distributions on C_i (i.e., the set of all **mixed** strategies of player i). We extend the payoff functions u_i from C to $S = S_1 \times ... \times S_N$ by expected utility. The normative assumptions justifying this type of preferences over uncertain outcomes are the subject of the next section.

In the resulting game S_i will be convex compact subsets of some finite-dimensional vector space. Extended payoff functions $u_i : S \to \mathbb{R}$ will be continuous on S, and $u_i(\cdot, s_{-i})$ will be be concave (actually, linear) on S_i. Thus we can apply the theorem above to show that

Theorem 6 Γ_f *always has a Nash equilibrium in mixed strategies.*

Note that a Nash equilibrium of the initial game remains an equilibrium in its extension to mixed strategies.

The Problems offer several applications of Nash's theorem, in particular problem 5.

2   Games with increasing best reply

A class of games closely related to dominance-solvable games consist of those where the best reply functions (or correspondences) are non decreasing. In those games existence of a Nash equilibrium is guaranteed by the general fixed point theorem of Tarski, stating that an increasing function in a lattice must have at least a fixed point.

A simple instance of this result is that any non decreasing function f from $[0,1]^n$ into itself (i.e., $x \leq x' \Rightarrow f(x) \leq f(x')$) has a fixed point. We also know that it has a smallest fixed point, and a largest fixed point. Now consider a symmetric game where $S_i = [0,1]$ and the (symmetric) best reply function $s \to br(s, \cdots, s)$ is non decreasing. This function must cross the diagonal, which shows that a symmetric Nash equilibrium exists. The next Proposition generalizes this observation.

Proposition 7 *Let the strategy sets S_i be either finite, or real intervals $[a_i, b_i]$. Assume the best reply functions in the game $\mathcal{G} = (N, S_i, u_i, i \in N)$ are single valued and non decreasing*

$$s_{-i} \leq s'_{-i} \Rightarrow br_i(s_{-i}) \leq br_i(s'_{-i}) \text{ for all } i \text{ and } s_{-i} \in S_{-i}$$

Then the game has a smallest Nash equilibrium outcome s_- and s_+ a largest one s_+. If a best reply dynamics starting from a converges, its limit is s_-; if a best reply dynamics starting form b converges, it is to s_+.

Proposition 8 *Say that the payoff functions u_i satisfy the single crossing property if for all i and all $s, s' \in S_N$ such that $s \leq s'$ we have*

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \Rightarrow u_i(s'_i, s'_{-i}) > u_i(s_i, s'_{-i})$$

$$u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}) \Rightarrow u_i(s'_i, s'_{-i}) \geq u_i(s_i, s'_{-i})$$

Under the SC property, define br_i^- and br_i^+ to be respectively the smallest and largest element of the best reply correspondence. They are both non-decreasing. The sequences s_-^t and s_+^t defined as

$$s_-^0 = a; s_-^{t+1} = br_i^-(s_-^t); s_+^0 = b; s_+^{t+1} = br_i^+(s_+^t)$$

are respectively non decreasing and non increasing, and they converge respectively to the smallest Nash equilibrium s_- and to the largest one s_+. Finally the successive elimination of strictly dominated strategies converges to $[s_-, s^+]$

$$\{s_-, s_+\} \subset \cap_{t=1}^{\infty} S_N^t \subset [s_-, s_+]$$

In particular if the game has a unique equilibrium outcome, it is strictly dominance-solvable.

Note that if u_i is twice differentiable the SC property holds if and only if

$$\frac{\partial^2 u_i}{\partial s_i \partial s_j} \geq 0 \text{ on } [a, b].$$

Example 1 *Voluntary contribution to a public good (continued)*
Consider Example 20 of chapter 2 where $z \rightarrow B(z)$ is convex over $\mathbb{R}_+$. Then the game has the SC property, therefore all the properties spelled above apply. As the game is also a potential game, we conclude that it is strictly dominance solvable if the potential function $P(s) = B(s_N) - \sum_i C_i(s_i)$ has a unique coordinate-wise maximum. An example is $B(x) = \frac{1}{2}x^2, C_i(x) = \frac{1}{3}\frac{x^3}{a_i^2}$.

Example 2 *A search game*
Each player exerts effort searching for new partners. The probability that player i finds any other player is $s_i, 0 \leq s_i \leq 1$, and when i and j meet, they derive the benefits α_i and α_j respectively. The cost of the effort is $C_i(s_i)$. Hence the payoff functions

$$u_i(s) = \alpha_i s_i s_{N \setminus \{i\}} - C_i(s_i) \text{ for all } i$$

Assuming only that C_i is increasing, we find that the game satisfies the single crossing property. The strategy profile $s_- = 0$ is always an equilibrium, and the largest equilibrium s_+ is Pareto superior to s_-.
The game is a potential game as well, provided we rescale the utility functions as

$$v_i(s) = \frac{1}{\alpha_i} u_i(s) = s_i s_{N \setminus \{i\}} - \frac{1}{\alpha_i} C_i(s_i)$$

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so the potential is

$$P(s) = \sum_{i \neq j} s_i s_j - \sum_i \frac{1}{\alpha_i} C_i(s_i)$$

Example 3 *price competition*
Each firm has a linear cost production (set to zero without loss of generality) and chooses a non negative price p_i. The resulting demand and net payoff for firm i are

$$D_i(p) = (A_i - \frac{\alpha_i}{3} p_i^2 + \sum_{j \neq i} \beta_j p_j)_+ \text{ and } u_i(p) = p_i D_i(p)$$

Check that for any p_{-i}, the best reply of player i is

$$br_i(s_{-i}) = \frac{1}{\sqrt{\alpha_i}} \sqrt{A_i + \sum_{j \neq i} \beta_j p_j}$$

so that the game has increasing best reply functions. On the other hand it does not have the single crossing property.
In the symmetric case $(A_i = A, \alpha_i = \alpha, \beta_i = \beta)$, one checks that its equilibrium is unique and is strongly stable.

3 Von Neumann Morgenstern utility

We axiomatize preferences over random outcomes represented by an expected utility function.
Notation:

C is the finite set of outcomes (consequences), $C = \{c_1, \cdots, c_m\}$

Δ is the set of lotteries on C with generic element $L = (p_1, \cdots, p_m), p_j \geq 0$ for all j and $\sum_1^m p_j = 1$

Definition 9 *(compound lottery) Given K (simple) lotteries $L_k \in \Delta, k = 1, \cdots, K$, and a probability distribution $\pi = (\pi_1, \cdots, \pi_K)$, the compound lottery $(L_k, k = 1, \cdots, K; \pi)$ is the random choice of an outcome in C where we pick first a lottery L_k according to π, then an outcome in C according to L_k.*

The simple lottery $L = \sum_1^K \pi_k L_k$ give the same ultimate probability distribution over outcomes as the compound lottery $(L_k, k = 1, \cdots, K; \pi)$, yet it is not unreasonable to distinguish these two objects from a decision-theoretic viewpoint.

Consequentialist axiom: *the preferences of our decision maker over a compound lottery do not distinguish it from the associated simple lottery.*

In view of this axiom, the preferences of our agent over the random outcomes in C, obtained via compound lotteries of arbitrary order, are represented by a rational preference (complete, transitive) $\preceq$ over Δ.

Continuity axiom: *upper and lower contour sets of $\preceq$ are closed in Δ.*

By the classic Debreu theorem, the continuity axiom implies that these preferences can be represented by a continuous utility function.

Independence axiom: *for all $L, L', L'' \in \Delta$, for all $\alpha \in [0,1]$*

$$L \succeq L' \Leftrightarrow \alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''$$

The independence axiom is very intuitive given consequentialism, and yet extremely powerful. It is the mathematical engine driving th VNM theorem.

Definition 10 *The utility function $U : \Delta \to \mathbb{R}$ has the Von Neumann Morgenstern expected utility form if there exists real numbers $u_1, \cdots, u_m$ such that*

$$U(L) = \sum_{j=1}^{m} u_j p_j \text{ for all } L = (p_1, \cdots, p_m) \in \Delta$$

An equivalent definition is that the function U is affine on Δ, namely

$$U(\alpha L + (1-\alpha)L') = \alpha U(L) + (1-\alpha)U(L') \text{ for all } L, L' \in \Delta, \text{ and all } \alpha \in [0,1]$$

An important invariance property of the VNM representation of a preference relation on Δ: if U has the VNM form and represents $\preceq$, so does $\beta U + \gamma$ for any numbers $\beta > 0$ and $\gamma \in \mathbb{R}$. Conversely, such utility functions are the only alternative VNM representations of $\preceq$.

A consequence of this invariance is that differences in cardinal utilities have meaning:

$$u_1 - u_2 > u_3 - u_4 \Leftrightarrow \frac{1}{2}u_1 + \frac{1}{2}u_4 > \frac{1}{2}u_2 + \frac{1}{2}u_3$$

Theorem 11 *(Von Neumann and Morgenstern) The preferences $\preceq$ over Δ meet the Continuity and Independence axioms if and only if they are representable in the expected utility form.*

A consequence of the Independence axiom is the property that indifference contours of these preferences are straight lines; this is the key argument in the proof of the Theorem.

Critique of the independence axiom: *the Allais paradox*

Consider three outcomes

- c_1: win a prize of 800K

- c_2: win a prize of 500K

- c_3: no prize.

Now consider the two choices between two pairs of lotteries

$$L_1 = (0, 1, 0) \text{ versus } L_1' = (0.1, 0.89, 0.01)$$

$$L_2 = (0, 0.11, 0.89) \text{ versus } L_2' = (0.1, 0, 0.9)$$

A commonly observed set of preferences are:

$$L_1 \succ L_1', \ L_2' \succ L_2$$

but these preferences are not compatible with VNM expected utility!

4 Mixed strategy equilibrium

Here we discuss a number of examples to illustrate both the interpretation and computation of mixed strategy equilibrium in n-person games. We start with *two-by-two games (*two players have two strategies each).

Example 4 *crossing games*
We revisit the example 12 from chapter 2

$$\begin{array}{ccc} stop & 1,1 & 1-\varepsilon,2 \\ go & 2,1-\varepsilon & 0,0 \\ & stop & go \end{array}$$

and compute the (unique) mixed strategy equilibrium

$$s_1^* = s_2^* = \frac{1-\varepsilon}{2-\varepsilon}\text{stop} + \frac{1}{2-\varepsilon}\text{go}$$

with corresponding utility $\frac{2-2\varepsilon}{2-\varepsilon} \simeq 1 - \frac{\varepsilon}{2}$ for each player. So an accident (both player go) occur with probability slightly above $\frac{1}{4}$. Both players enjoy an expected utility only slightly above their secure (guaranteed) payoff of $1-\varepsilon$. Under s_1^*, on the other hand, player 1 gets utility close to $\frac{1}{2}$ about half the time: for a tiny increase in the expected payoff, our player incur a large risk. [Note that this is a critique of the VNM utility representation, not of the mixed strategy equilibrium concept.]

The point is stronger in the following variant of the crossing game

$$\begin{array}{ccc} stop & 1,1 & 1+\varepsilon,2 \\ go & 2,1+\varepsilon & 0,0 \\ & stop & go \end{array}$$

where the (unique) mixed strategy equilibrium is

$$s_1^* = s_2^* = \frac{1+\varepsilon}{2+\varepsilon}\text{stop} + \frac{1}{2+\varepsilon}\text{go}$$

and gives to each player exactly her guaranteed utility level in the mixed game. Indeed a (mixed) prudent strategy of player 1 is

$$\widetilde{s}_1 = \frac{2}{2+\varepsilon}\text{stop} + \frac{\varepsilon}{2+\varepsilon}\text{go}$$

and it guarantees the expected utility $\frac{2+2\varepsilon}{2+\varepsilon}$, which is also the mixed equilibrium payoff. Now the case for playing the equilibrium strategy in lieu of the prudent one is even weaker, unless we maintain a strict interpretation of the VNM preferences.

Computing the mixed equilibrium or equilibria of a finite n-person game follows the same general approach as for two-person zero-sum games. Here too the difficulty is to identify the support of the equilibrium strategies. In a two-person games, we can always find at least one equilibrium with two supports

of equal sizes, but this is not true any more with three or more players (for instance player 3 may have a dominant strategy, while players 1 and 2 play mixed strategies of equal size0. Once this is done we need to solve a system of linear equalities and inequalities.

Unlike in two-person zero-sum games, we may have several mixed equilibria with different payoffs. A deep theorem shows that for "most games", the number of mixed or pure equilibria is *odd*.

Example 5 *public good provision (Bliss and Nalebuff)*
Each one of the n players can provide the public good (hosting a party, slaying the dragon, or any other example where only one player can do the job) at a cost $c > 0$. The benefit is b to every agent if the good is provided. We assume $c < nb$: the social benefit justifies providing the good. The players can divide the burden of providing the good by the following use of lotteries. Each player chooses to step forward (volunteer) or not. If nobody volunteers, the good is not provided; if some players volunteer, we choose one of them with uniform probability to provide the good.

If $b < c$, the game in pure strategies is a classic Prisoner's Dilemma (section 2.2.3) . If $b > c$, it resembles the war of attrition (sectione 2.2.1) in that we have n pure strategy equilibria where one player provides the good and the other free ride.

The game is symmetrical so we look for a symmetrical equilibrium in mixed strategies in which every player steps forward with probability $p^*, 0 < p^* < 1$. Then each player is indifferent between stepping forward or not. The latter gives the expected utility $b(1 - (1-p)^{n-1})$. The former gives the utility

$$b - c(\sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{k+1} p^k (1-p)^{n-1-k}) = b - c\frac{1 - (1-p)^n}{np}$$

(because $\frac{\binom{n-1}{k}}{k+1} = \frac{\binom{n}{k+1}}{n}$). Therefore p^* solves

$$\frac{nb}{c}p = \frac{1 - (1-p)^n}{(1-p)^{n-1}} = f(p)$$

Notice that f is convex, increasing, from $f(0) = 0$ to $f(1) = \infty$, and $f'(0) = n$. Therefore if $b < c$, the only solution of the equation above is $p = 0$ and we are back to the Prisoner's Dilemna. But if $b > c$, there is a unique equilibrium in mixed strategies. For instance if $n = 2$, we get

$$p_2^* = \frac{2(b-c)}{2b-c} \text{ and } u_i(p^*) = \frac{2b(b-c)}{2b-c}$$

One checks that as n grows, p_n^* goes to zero as $\frac{K}{n}$ where K is the solution of

$$\frac{c}{b} = \frac{Ke^K}{1 - e^{-K}}$$

therefore the probability that the good be provided goes to $1 - e^{-K}$, but the probability of volunteering of each player goes to zero.

Note that, because $p^*(k)$ is decreasing in k, the game has many other equilibria, where only a subset of k players step forward with the corresponding probability $p^*(k)$.

Infinite sets of pure strategies

Existence of a Nash equilibrium in mixed strategies holds under the same assumptions as Glicksberg theorem for two person zero-sum games, namely strategy sets are convex and compact, and utility functions are continuous. Here is an example.

Example 6 *lobbying game (a.k.a. all-pay first price auction)*
The n players compete for a prize of \$$p$ and can spend \$$s_i$ on lobbying (bribing) the relevant jury members. The largest bribe wins the prize; all the money spent on bribes is lost to the players. Hence the payoff functions

$$u_i(s) = p - s_i \text{ if } s_i > \max_{j \neq i} s_j; = -s_i \text{ if } s_i < \max_{j \neq i} s_j; = \frac{p}{K} - s_i \text{ if } s_i = \max_{j \neq i} s_j$$

The strategies $s_i > p$ are dominated by the null strategy. But the game has no equilibrium in pure strategies. In the symmetrical mixed Nash equilibrium each player independently chooses a bid in $[0, p]$ according to the cumulative distribution function F. As in the previous example we compute the expected payoff to player 1 using his pure strategy s_1 against the mixed strategy of everyone else: $(p - s_1)F^{n-1}(s_1) - s_1(1 - F^{n-1}(s_1))$. That this payoff is independent of $s_1 \in [0, p]$ gives

$$F(x) = \left(\frac{x}{p}\right)^{\frac{1}{n-1}}$$

As in the above example the equilibrium payoff is zero, just like the guaranteed payoff from a null bid.

Example 7 *war of attrition (a.k.a. all-pay second price auction)*
We revisit the game of timing in Example 7 Chapter 2, specifying VNM utilities. The n players compete for a prize worth \$$p$ by "hanging on" longer than everyone else. Hanging on costs \$1 per unit of time. Once a player is left alone, he wins the prize without spending any more effort.

$$u_i(s) = p - \max_{j \neq i} s_j \text{ if } s_i > \max_{j \neq i} s_j; = -s_i \text{ if } s_i < \max_{j \neq i} s_j; = \frac{p}{K} - s_i \text{ if } s_i = \max_{j \neq i} s_j$$

where K is the number of largest bids.

One checks first that no pure strategy is dominated. In addition to the pure equilibria described in Example 7, Chapter 2, we have one symmetrical equilibrium in completely mixed strategies where each player independently chooses s_i in $[0, \infty[$ according to a cumulative distribution function F: so $F(t) = proba\{s_i \leq t\}$. To compute F we assume that all players $2, \cdots, n$ choose s_i according to F and consider the expected payoff of player 1 using the pure strategy s_1:

$$\int_0^{s_1} (p - t)G'(t)dt - s_1(1 - G(s_1)), \text{ where } G(t) = F^{n-1}(t) = proba\{\max_{j \neq i} s_j \leq t\}$$

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Then we write that all pure strategies s_1 give the same payoff to player 1, i.e., the above expression is constant in s_1. This gives $pG'(t) + G(t) = 1$ for all t; taking the initial condition $G(0) = 0$ into account, we find

$$F(x) = (1 - e^{-\frac{x}{p}})^{\frac{1}{n-1}}$$

In particular the support of this distribution is $[0, \infty[$ and for any $B > 0$ there is a positive probability that a player bids above B. The payoff to each player is zero so the mixed strategy is not better than the prudent one (zero bid) payoffwise. It is also more risky. There is no equilibrium in this game where players use strategies with bounded support.

5 Correlated equilibrium

Given a finite n-players game in strategic form $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, a correlation device is a *lottery* L over the set $C = C_1 \times ... \times C_n$ of strategy profiles. The interpretation is that the lottery itself is a *non binding* agreement to play according to its outcome. Thus the lottery is built jointly by the players (much like we say that the players jointly reach an agreement to play a certain Nash equilibrium), and once it draws an outcome $x \in C$, the players are *supposed to* play accordingly, namely player i chooses x_i in C_i.

If the outcome of the lottery is publicly known, the agreement will be self enforcing if and only if the support of the lottery consists of Nash equilibrium outcomes (in pure strategies). Then the lottery is a simple coordination device over a set of equilibria in pure strategies. This is a useful coordination device, for instance to achieve a fair compromise between asymmetric equilibria in a symmetric game. In the crossing game of example 1, tossing a fair coin between the two equilibria yields a payoff of $1.5 \pm \frac{\varepsilon}{2}$, much better than the payoff of the only symmetric equilibrium, in mixed strategies. We can interpret a red light as achieving precisely this kind of coordination when two lines of traffic cross. Another example is the war of attrition (Example 7), where the players can coordinate on a fair compromise between two Pareto optimal equilibria in pure strategies.

More interesting is the scenario where the distribution L is known to everyone, but the outcome of the lottery is only *partially revealed* to each player. Specifically player i learns the i-th coordinate of the outcome x, but no more: then she evaluates the random strategies chosen by other players according to the conditional probability of L given x_i. If other players are indeed following the recommendation of the correlation device, this evaluation is correct. Now the equilibrium (self-enforcing) property of the lottery L states that player i's best reply to any recommendation x_i is to comply.

Given a lottery $L \in \Delta(C)$ we write its support $[L] \subset C$ and the projection of the support on C_i as $proj_i\{[L]\}$. This set contains the strategies of player i that the device recommends to play with positive probability. For any i and $x_i \in C_i$, we denote by $L(x_i)$ the corresponding conditional probability of L on

$C_{N \setminus \{i\}}$. Thus if L_x denotes the probability that L selects outcome x, we have

$$L(x_i)_{x_{-i}} = \frac{L_{(x_i, x_{-i})}}{\sum_{y_{-i} \in C_{N \setminus \{i\}}} L_{(x_i, y_{-i})}} \text{ for all } x_i \in proj_i\{[L]\} \text{ all } x_{-i} \in C_{N \setminus \{i\}}$$

Definition 12 *A lottery $L \in \Delta(C)$ is a correlated equilibrium of the game $(N, (C_i)_{i \in N}, (u_i)_{i \in N})$ if for all $i \in N$ we have*

$$u_i(x_i, L(x_i)) \geq u_i(y_i, L(x_i)) \text{ for all } y_i \in C_i \text{ and all } x_i \in proj_i\{[L]\}$$

$$\Leftrightarrow \sum_{y_{-i} \in C_{N \setminus \{i\}}} u_i(x_i, y_{-i}) L_{(x_i, y_{-i})} \geq \sum_{y_{-i} \in C_{N \setminus \{i\}}} u_i(y_i, y_{-i}) L_{(x_i, y_{-i})} \text{ for all } y_i, x_i \in C_i$$

If $s \in \Delta(C_1) \times ... \times \Delta(C_n)$ is an equilibrium in mixed strategies, then the lottery $L = s_1 \oplus s_2 \oplus \cdots \oplus s_n$ is a correlated equilibrium. This remark establishes that a correlated equilibrium always exists in a finite game.

The most important feature of the set $\mathcal{C}$ of correlated equilibria is that it is a convex, compact subset of $\Delta(C)$. Indeed $\mathcal{C}$ is defined by a finite set of linear inequalities in $\Delta(C)$. Thus it contains all convex combinations of Nash equilibria, pure and mixed.

In some games, that is all. For instance suppose each player has a strictly dominant strategy: then the unique Nash equilibrium is also the unique correlated equilibrium. Indeed the support of any correlated equilibrium must resist the successive elimination of strictly dominated strategies. Furthermore, there is always one correlated equilibrium of which the support resists the successive elimination of weakly dominated strategies.

But as soon as we have several Nash equilibria (pure or mixed) not in a rectangular position, there are more correlated equilibria. In some games this only helps to average between pure equilibria, as in Example 4 above. In other games, correlation allows a considerable improvement upon the Nash equilibrium outcomes.

Example 8 *another Battle of the Sexes*

	home	theater
home	10, 10	5, 13
theater	13, 5	0, 0

One of the spouses must stay home, lest they are both very unhappy to call for a baby sitter. Both would prefer to go to the theater if the other stays home. Each must commit to one of the two strategies *before* returning home, and without the possibility to communicate with each other.

There are two equilibria in pure strategies, and a mixed equilibrium where each player goes out with probability $\frac{3}{8}$. The expected payoff of the latter is 8.1 for each. Tossing a fair coin *before leaving to work* between the two equilibria yields the payoff 9 for each spouse.

There is a better correlated equilibrium, choosing (theater, home) and (home, theater) each with probability $\frac{3}{11}$, and (home,home) with probability $\frac{5}{11}$. The expected payoff is now 9.45 for each.

Example 9 *musical chairs*

We have n players and 2 "chairs" (locations), with $n > 5$. The game is symetrical. Each player chooses a chair. His payoff is $+4$ if he is alone to make this choice, 1 if one other player (exactly) makes the same choice, and 0 otherwise (i.e., if his choice is shared by at least 2 other players).

In a pure strategy equlibria of the game, each chair is filled by two or more players and all such outcomes are equilibria. The total payoff is 2 or 0. In the symmetric mixed equilibrium each player chooses a chair with probability 0.5, and the resulting expected payoff is

$$4\frac{1}{2^n} + 1\frac{n-1}{2^n} = \frac{2n-1}{2^n} \ll 2$$

(there are no other mixed equilibria)

The best symmetric correlated equilibrium (i.e., the one giving the highest total payoff) selects with probability $\pi = \frac{2}{n-3}$ a distribution where one player sits alone (and chooses with uniform probability among all such distributions), and with probability $1 - \pi = \frac{n-5}{n-3}$ it picks a distribution where two players share one chair (and chooses with uniform probability among all such distributions). The total payoff is $2 + \frac{4}{n-3}$.

6 Games of incomplete information

A game in Bayesian form(or Bayesian game) specifies

- the set N of players

- the set of pure strategies X_i for each player i

- the set of *types* T_i of each player i

- the set of *beliefs* of each player i, represented by a probability distribution $\pi_i(\cdot|t_i)$ over $T_{N \setminus \{i\}}$: one distribution for each possible type of player i

- the payoff function $u_i(x,t)$ for each player i, where $x \in X_N$ and $t \in T_N$.

A Bayesian equilibrium is decribed by a mixed strategy for each player, conditional on his type: $s_i(t_i) \in \Delta(X_i)$. The equilibrium property is

$$\forall i, t_i \in T_i, \forall s_i' \in \Delta(X_i):$$

$$\sum_{t_{-i} \in T_{N \setminus \{i\}}} \pi_i(t_{-i}|t_i) u_i(s(t), t) \geq \sum_{t_{-i} \in T_{N \setminus \{i\}}} \pi_i(t_{-i}|t_i) u_i(s_i'; s_{-i}(t_{-i}), t)$$

where we use the notation

$$s(t) \in \Pi_{i \in N} \Delta(X_i), s_{-i}(t_{-i}) \in \Pi_{j \in N \setminus \{i\}} \Delta(X_j): s_j(t) = s_j(t_j)$$

It is enough in the equilibrium property to consider deviations to pure strategies $x_i \in X_i$. Therefore the number of inequalities characterizing the equilibrium is $\sum_i |T_i||X_i|$.

Theorem: *If the sets X_i and T_i are finite, the game possesses at least one Bayesian equilibrium.*

This is a direct consequence of Nash's theorem, after observing that a Bayesian equilibrium is a Nash equilibrium (in pure strategies) of the game with $\mathcal{N} = \oplus_i T_i$, strategy set $\Delta(X_i)$ for each player $(i, t_i) \in \mathcal{N}$ and payoffs

$$\widetilde{u}_{(i,t_i)}(s) = \sum_{t_{-i} \in T_{N \setminus \{i\}}} \pi_i(t_{-i}|t_i) u_i(s_{(i,t_i)}; s_{(j,t_j)} \; j \in N \setminus \{i\})$$

This game meets all the assumptions of Nash's Theorem (in particular utility is linear in own strategy).

The common prior, common knowledge assumption

In most examples , the individual beliefs are consistent, they are derived from a common prior, namely a probability distribution π over T_N, and each player i learns her own type t_i. Thus player i's beliefs are described by the conditional probability $\pi_i(\cdot|t_i) = \pi(\cdot|t_i)$ of π upon learning one's type. This distribution π is *common knowledge*, which means that player i knows it, i knows that player j knows it, j knows that player i knows that player j knows it, and so on. More generally, for any sequence $i, j, k, \cdots, l$ of players (possibly with repetition): i knows that j knows that k knows that $\cdots$ that l knows it.

The classic story of the 40 prisoners illustrates the subtle role of the common knowledge assumption. Each has a white or black dot painted on his forehead. They see everyone else's dot, but they cannot talk to one another. Every morning they can approach the prison's warden: anyone who tells his own color is freed at once; if he is wrong he is immediately execcuted, so no prisoner will tell his own color unless he is absolutely certain.

All prisoners have a black dot, and nothing happens for a long time. One day the warden gathers them and says: there is at least one black dot among you (a fact they all already know). Forty days later, they all go to see the warden and all are freed.

In a Bayesian game where the beliefs are not consistent, the interpretation of the equilibrium notion is more difficult. Consider for instance a 2×2 two-person zero-sum game where if $t_1 = t_2$ the game has a value of $+1$, whereas if $t_1 \neq t_2$ the value is -1. If player 1 (resp. player 2) believes $t_1 = t_2$ (resp. $t_1 \neq t_2$) for sure, both players, "win" ex ante.

Example 10-a

Two players, player 1's type is known, that of player 2 is t_1 with probability 0.6, t_2 with probability 0.4:

	T	$1,2$	$0,1$		T	$1,3$	$0,4$
	B	$0,4$	$1,3$		B	$0,1$	$1,2$
	t_1	L	R		t_2	L	R

Player 2 has a dominant strategy, hence the unique equilibrium is in pure strategies:

$$x_1 = T; x_2 = L \text{ if } t_1, \ = R \text{ if } t_2$$

Note that this is not the same as playing the unique Bayesian equlibrium in each matrix separately, which makes no sense given player 1's information. Instead, player 1 compares

$$(0.6)u_1(T, (L, t_1)) + (0.4)u_1(T, (L, t_1)) \text{ and } (0.6)u_1(B, (L, t_1)) + (0.4)u_1(B, (L, t_1))$$

Example 10-b Same information structure, and the payoffs are now:

	T	$0,2$	$2,0$		T	$1,1$	$5,0$
	B	$2,0$	$0,2$		B	$0,5$	$3,3$
	t_1	L	R		t_2	L	R

Here the game under t_1 is essentially matching pennies, and under t_2 player 2 has a dominant strategy to play L. There is no pure strategy equilibrium, as the sequences of best replies are: $LL \to B \to RL \to T \to LL$, and $RR \to T$, $LR \to B$.

Mixed strateges for the two players are

$$s_1 = \lambda T + (1 - \lambda)B; \quad s_2 = \mu L + (1 - \mu)R \text{ if } t_1; = L \text{ if } t_2 \qquad (1)$$

(because under t_2 player 2 plays L for sure). In equilibrium, each player must make the other indifferent between his two pure strategies. Therefore player 1's mixed strategy is the optimal play for matching pennies, and player 2's strategy is computed as:

$$s_1^* = \frac{1}{2}T + \frac{1}{2}B; s_2^* = \frac{2}{3}L + \frac{1}{3}R \text{ if } t_1, \ = L \text{ if } t_2$$

Example 10-c Same information structure:

	T	$0,2$	$2,0$		T	$2,0$	$1,2$
	B	$2,0$	$0,2$		B	$0,3$	$2,0$
	t_1	L	R		t_2	L	R

Here again we have no pure strategy equilibrium, as the best reply sequence is $T \to LR \to B \to RL \to T$, and $LL \to B, RR \to T$. Bayesian strategies take the form (1) as above. Note that player 2 no longer has a dominant strategy in any of her types, and cannot be indifferent between her two pure choices for each one of her types. So she can only be indifferent at one of her types, and there are two possibilities:

1) player 1 makes player 2 indifferent at t_2, not at t_1. Then $s_1 = \frac{3}{5}T + \frac{2}{5}B$ and $s_2 = L$ if $t_1, \ = \mu L + (1 - \mu)R$ if t_2. We must choose μ to make player 1 indifferent, namely

$$(0.6)0 + (0.4)(2\mu + (1 - \mu)) = (0.6)2 + (0.4)(0\mu + 2(1 - \mu)) \Rightarrow \mu = \frac{4}{3}$$

contradiction!

2) player 1 makes player 2 indifferent at t_1, not at t_2. Then $s_1 = \frac{1}{2}T + \frac{1}{2}B$ and $s_2 = \mu L + (1 - \mu)R$ if t_1, $= L$ if t_2. Chosing μ to make player 1 indifferent gives

$$(0.6)2(1 - \mu) + (0.4)2 = (0.6)2\mu + (0.4)0 \Rightarrow \mu = \frac{5}{6}$$

and this is the unique Bayesian equilibrium of the game.

Example 11 *a two-person zero sum betting game*

Bob (column player) draws a card High or Low with equal probability $\frac{1}{2}$. Ann (row player) has a Medium card (a fact known to Bob). Bob can raise (R) or stay put (P). After seeing Bob's move, Ann can see (S) or fold (F). Payoffs are as follows

S	$-10, 10$	$-4, 4$		S	$10, -10$	$4, -4$
F	$-1, 1$	$1, -1$		F	$-1, 1$	$1, -1$
$High$	R	P		Low	R	P

Here Ann has 4 pure strategies denoted XY for do X if Bob raises, do Y if he does not; Bob's strategy depends on his type, and is written similarly XY for do X if High, do Y if Low (note the difference in interpretations).

Check first there is no pure strategy equilibrium, as the sequence of best replies is

$$\text{Bob}RR \rightarrow \text{Ann}SS; \text{Ann}SS \text{ or } SF \rightarrow \text{Bob}RP(\text{revealing}) \rightarrow \text{Ann}FS \rightarrow \text{Bob}RR$$

and similarly

$$\text{Bob}PR(\text{bluffing}) \rightarrow FS \rightarrow RR \rightarrow \cdots ; PP \rightarrow FF \rightarrow RR \rightarrow \cdots$$

Bob has a dominant strategy to raise if his card is high. Thus his P strategy reveals to Ann that he is Low, in which case she wants to see. Therefore the Bayesian equilibrium takes the form

Ann: $\quad p\delta_S + p'\delta_F$ if Bob raises; S if Bob stays put

Bob: $\quad R$ if High; $q\delta_R + q'\delta_P$ if Low

The equilibrium conditions are

for Ann: $\quad \dfrac{1}{1+q}(-10) + \dfrac{q}{1+q}(10) = -1 \Rightarrow q = \dfrac{9}{11}$

for Bob: $\quad p(-10) + p'(1) = -4 \Rightarrow p = \dfrac{5}{11}$

In equilibrium Ann expects to pay $\$\frac{6}{11}$ to Bob: private information is more valuable than second move.

Example 12: *first price auction (Vickrey)*

Pure strategies and payoffs are identical to those in example 12 Chapter 3. We have n players and one object. Each bids for the object, highest winner wins the object and pays own bid.

Each player draws a valuation in the $[0, 100]$ interval. The draws are IID with cumulative distribution function F. We assume that F is continuous: the underlying distribution has no atoms.

The *symmetrical* equilibrium has player i bid $x(t_i)$ where t_i is his (privately known) valuation. The expected payoff to player i from bidding y, given that other players use the equilibrium strategy $x(\cdot)$ is

$$u_i(y|t_i) = (t_i - y)\pi\{x(T_j) < y \text{ for all } j \neq i\}$$

where T_j is the type of player j, a random variable. Player i chooses his bid $y = x(t)$ so as to maximize $(t_i - x(t))proba\{x(T_j) < x(t) \text{ for all } j \neq i\}$. The equilibrium property is that $t = t_i$ is such a maximizer.

Check first that $x(\cdot)$ must be increasing. Fix $t, t', t < t'$, and set $p = \pi\{x(t_j) < x(t) \text{ for all } j \neq i\}$, $p' = \pi\{x(t_j) < x(t') \text{ for all } j \neq i\}$. The equilibrium conditions at t and t' give respectively

$$\{(t - x(t))p > (t - x(t')p', \text{ and } (t' - x(t'))p' > (t' - x(t)p\} \Rightarrow (t' - t)(p' - p) \geq 0$$

and the desired conclusion. Similar arguments show that $x(\cdot)$ must be continuous and differentiable.

We see now that the event $\{x(T_j) < x(t) \text{ for all } j \neq i\}$ is $\{T_j < t \text{ for all } j \neq i\}$ therefore $proba\{x(T_j) < x(t) \text{ for all } j \neq i\} = F^{n-1}(t)$.

It remains to write that $z \to (t - x(z))F^{n-1}(z)$ reaches its maximum at t, for all t. Differentiating:

$$x'(t)F^{n-1}(t) - (t - x(t))\{F^{n-1}(t)\}' = 0$$

The boundary condition is $x(0) = 0$. A zero valuation player does not want to bid any positive amount. The differential equation writes

$$\{x(t)F^{n-1}(t)\}' = t\{F^{n-1}(t)\}'; \quad x(0) = 0, \text{ and } F^{n-1}(0) = 0$$

Therefore

$$x(t) = \frac{\int_0^t z dF^{n-1}(z)}{F^{n-1}(t)} = E[t_{(2)}|t_{(1)} = t] \tag{2}$$

where $t_{(k)}$ is the k-th order statistics of the n variables t_i. To check the second equality, observe that for all $a, t, a < t$

$$\pi\{t_{(2)} \leq a|t_{(1)} = t\} = \pi\{t_{-1} \leq a|t_{-1} \leq t; t_1 = t\} = \pi\{t_{-1} \leq a|t_{-1} \leq t\} = \frac{F^{n-1}(a)}{F^{n-1}(t)}$$

(where the first equality follows from the fact that types are identically distributed, and the second from the fact they are stochastically independent).

Equation (2) says that the equilibrium bid is the expected value of the second highest bid, conditional on your own bid winning the object.

For instance assume the uniform distribution on $[0, 100]$, so that $F(t) = t$, then $x(t) = \frac{n-1}{n}t$ and the expected highest bid (revenue of the seller) is

$$E[x(t_{(1)})] = \frac{n-1}{n}E[t_{(1)}] = \frac{n-1}{n+1}100$$

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Moreover the efficient buyer (the one with the highest valuation) gets the object, therefore the expected joint surplus to the seller and bidders is $E[t_{(1)}] = \frac{n}{n+1}100$. This leaves only an expected gain of $\frac{1}{n(n+1)}100$ per bidder!

Interestingly this sharing of the surplus between buyers and the seller is the same as in Vickrey's second price auction, because there the revenue of the seller is

$$E[t_{(2)}] = \int_0^{100} E[t_{(2)}|t_{(1)} = t]dF^n(t) = \int_0^{100} x(t)dF^n(t) = E[x(t_{(1)})]$$

Note that there are (many0 other equilibria in which the players use different bidding strategies. Describing them all is an open question.

Example 13 *sealed bids double auction (Myerson and Satterthwaite)*

The object is worth a to the seller, b to the buyer. Both a and b are IID on $[0, 300]$ with uniform distribution. They play the *sealed bid double auction* game: they independently and simulatneously send an ask price x (seller) and an offer price y (buyer). If $x > y$, no trade takes place; if $x \le y$, trade takes place at price $p = \frac{x+y}{2}$.

One checks first that $x(a) = a, y(b) = b$ is not an equilibrium. Suppose the seller plays $x(a) = a$, and the buyer is of type b; his profit $\int_0^y (b - \frac{y+a}{2})db$ is maximized at $y = \frac{2}{3}b$.

We compute the *linear equilibrium*, where each player uses a bid function that is linear in own valuation

$$x(a) = \alpha a + \beta; y(b) = \gamma b + \delta \tag{3}$$

We compute the best reply functions of our two players.

If the seller uses $x(\cdot)$ in (3), the expected profit of a type b buyer offering y is

$$\int_0^{\frac{y-\beta}{\alpha}} (b - \frac{y + \alpha a + \beta}{2})da = (\frac{y-\beta}{\alpha})(b - \frac{y+\beta}{2}) - \frac{\alpha}{4}(\frac{y-\beta}{\alpha})^2 = (\frac{y-\beta}{\alpha})(b - \frac{\beta}{4} - \frac{3}{4}y)$$

maximized at $y(b) = \frac{2b+\beta}{3}$.

If the buyer uses $y(\cdot)$ in (3), trade will occur if the seller's offer x is such that $x \le y(b) \Leftrightarrow b \ge \frac{x-\delta}{\gamma}$. The expected profit of a type a seller offering x is

$$\int_{\frac{x-\delta}{\gamma}}^{300} (\frac{x + \gamma b + \delta}{2} - a)db = \frac{1}{2\gamma}\{-\frac{3}{2}x^2 + (300\gamma + 2a + \delta)x + \text{ constant}\}$$

It is maximized at $x(a) = \frac{1}{3}(2a + 300\gamma + \delta)$.

Thus the unique candidate linear equilibrium is

$$x(a) = \frac{2}{3}a + 75; y(b) = \frac{2}{3}b + 25$$

It remains to check that participation is voluntary, i.e., no one would prefer to abstain from bidding. A buyer of type $b < 75$ bids *above* his own valuation,

$y(b) > b$, but as the seller's offer is never below \$75, such an offer is never accepted. Similarly a seller of type $a > 225$ bids $x(a) < a$, but again, this offer is irrelevant as $y(b) \leq 225$ for all b.

Finally we compute the welfare loss at this equilibrium. Trade occurs only if $x(a) \leq y(b) \Leftrightarrow b \geq a + 75$. Therefore the loss is

$$\frac{1}{300^2} \int \int_{a \leq b \leq a+75} (b-a)dadb = \frac{125}{16} \simeq 7.8$$

so about 16% of the efficient expected surplus

$$\frac{1}{300^2} \int \int_{a \leq b} (b-a)dadb = 50$$

It is important to keep in mind that the liner equilibrium is but one equilibrtium among many others, non linear equilibria. Computing all equilibria of the double auction game is an open problem. See Problem 20 for a family of very simple "fixed price equilibria", and Problem 21 for alternative trade mechanisms in the same context.

7   Problems for Chapter 4

Problem 1
a) In the two-by-two game

$$
\begin{array}{c c c}
T & 5,5 & 4,10 \\
B & 10,4 & 0,0 \\
 & L & R
\end{array}
$$

Compute all Nash equilibria. Show that a slight *increase* in the (B, L) payoff to the row player results in a *decrease* of his mixed equilibrium payoff.
b) Consider the crossing game of example 4

$$
\begin{array}{c c c}
stop & 1,1 & 1-\varepsilon, 2 \\
go & 2, 1-\varepsilon & 0,0 \\
 & stop & go
\end{array}
$$

and its variant where strategy "go" is more costly by the amount $\alpha, \alpha > 0$, to the row player:

$$
\begin{array}{c c c}
stop & 1,1 & 1-\varepsilon, 2 \\
go & 2-\alpha, 1-\varepsilon & -\alpha, 0 \\
 & stop & go
\end{array}
$$

Show that for α and ε small enough, row's mixed equilibrium payoff is *higher* if the go strategy is more costly.

Problem 2

Three plants dispose of their water in the lake. Each plant can send clean water ($s_i = 1$) or polluted water ($s_i = 0$). The cost of sending clean water is c. If only one firm pollutes the lake, there is no damage to anyone; if two or three firms pollute, the damage is a to everyone, $a > c$.

Compute all Nash equilibria in pure and mixed strategies.

Problem 3

Give an example of a two-by-two game where no player has two equivalent pure strategies, and the set of Nash equilibria is infinite.

Problem 4

A two person game with finite strategy sets $S_1 = S_2 = \{1, \cdots, p\}$ is represented by two $p \times p$ payoff matrices U_1 and U_2, where the row player is labeled 1 and the column player is 2. The entry $U_i(j, k)$ is player i's payoff when row chooses j and column chooses k. Assume that both matrices are invertible and denote by $|A|$ the determinant of the matrix A. Then write $\widetilde{U}_i(j, k) = (-1)^{j+k} |U_i(j, k)|$ the (j, k) cofactor of the matrix U_i, where $U_i(j, k)$ is the $(p-1) \times (p-1)$ matrix obtained from U_i by deleting the j row and the k column.

Show that if the game has a completely mixed Nash equilibrium, it gives to player i the payoff

$$\frac{|U_i|}{\sum_{1 \leq j, k \leq p} \widetilde{U}_i(j, k)}$$

Problem 5

In this symmetric two-by-two-by-two (three-person) game, the mixed strategy of player i takes the form $(p_i, 1 - p_i)$ over the two pure strategies. The resulting payoff to player 1 is

$$u_1(p_1, p_2, p_3) = p_1 p_2 p_3 - 3 p_1 (p_2 + p_3) + p_2 p_3 - p_1 - 2(p_2 + p_3)$$

Find the symmetric mixed equilibrium of the game. Are there any non symmetric equilibria (in pure or mixed strategies)?

Problem 6

Let $(\{1, 2\}, C_1, C_2, u_1, u_2)$ be a finite two person game and $\mathcal{G} = (\{1, 2\}, S_1, S_2, u_1, u_2)$ be its mixed extension. Say that the set $\mathcal{NE}(\mathcal{G})$ of mixed Nash equilibrium outcomes of $\mathcal{G}$ has the *rectangularity property* if we have for all $s, s' \in S_1 \times S_2$

$$s, s' \in \mathcal{NE}(\mathcal{G}) \Rightarrow (s'_1, s_2), (s_1, s'_2) \in \mathcal{NE}(\mathcal{G})$$

a) Prove that $\mathcal{NE}(\mathcal{G})$ has the rectangularity property if and only if it is a convex subset of $S_1 \times S_2$.
b) In this case, prove there exists a Pareto dominant mixed Nash equilibrium s^*:

$$\text{for all } s \in \mathcal{NE}(\mathcal{G}) \Rightarrow u(s) \leq u(s^*)$$

Problem 7 *all-pay second price auction*

This is a variant of example 6 with only two players who value the prize respectively at a_1 and a_2. The payoff are

$$u_i(s_1, s_2) = a_i - s_j \text{ if } s_j < s_i; = -s_i \text{ if } s_i < s_j; = \frac{1}{2} a_i - s_i \text{ if } s_j = s_i;$$

For any two numbers b_1, b_2 in $[0,1]$ such that $\max\{b_1, b_2\} = 1$, consider the mixed strategy of player i with cumulative distribution function

$$F_i(x) = 1 - b_i e^{-\frac{x}{a_j}}, \text{ for } x \geq 0$$

Show that the corresponding pair of mixed strategies (s_1, s_2) is an equilibrium of the game.

Riley shows that these are the only mixed equilibria of the game.

Problem 8 *all-pay first price auction*

This is a variant of Example 7 with only two players who value the prize respectively at a_1 and a_2. The payoffs are

$$u_i(s_1, s_2) = a_i - s_i \text{ if } s_j < s_i; = -s_i \text{ if } s_i < s_j; = \frac{1}{2}a_i - s_i \text{ if } s_j = s_i$$

Assume $a_1 \geq a_2$. Show that the following is an equilibrium:

player 1 chooses in $[0, a_2]$ with uniform probability;

player 2 bids zero with probability $1 - \frac{a_2}{a_1}$, and with probability $\frac{a_2}{a_1}$ he chooses in $[0, a_2]$ with uniform probability.

Riley shows this is the unique equilibrium if $a_1 > a_2$.

Problem 9 *first price auction*

This is a variant of Example 12 Chapter 2 where the two players value the prize respectively at a_1 and a_2. Each player bids $\$s_i$, where $s_i \in \mathbb{R}_+$ (instead of integers in Example 12, Chapter 2).The payoffs are

$$u_i(s_1, s_2) = a_i - s_i \text{ if } s_j < s_i; = 0 \text{ if } s_i < s_j; = \frac{1}{2}(a_i - s_i) \text{ if } s_j = s_i$$

a) Assume $a_1 = a_2$. Show that the only Nash equilibrium of the game in mixed strategies is $s_1 = s_2 = a_i$.

b) Assume $a_1 > a_2$. Show there is no equilibrium in pure strategies. Show that in any equilibrium in mixed strategies

player 1 bids a_2

player 2 chooses in $[0, a_2]$ according to some probability distribution π such that for any interval $[a_2 - \varepsilon, a_2]$ we have $\pi([a_2 - \varepsilon, a_2]) \geq \frac{\varepsilon}{a_2 - a_1}$.

Give an example of such an equilibrium.

Problem 10 *a location game*

Two shop owners choose the location of their shop in $[0,1]$. The demand is inelastic; player 1 captures the whole demand if he locates where player 2 is, and player 2's share increases linearly up to a cap of $\frac{2}{3}$ when he moves away from player 1. The sets of pure strategies are $C_i = [0,1]$ and the payoff functions are:

$$u_1(x_1, x_2) = 1 - |x_1 - x_2|$$
$$u_2(x_1, x_2) = \min\{|x_1 - x_2|, \frac{2}{3}\}$$

a) Show that there is no Nash equilibrium in pure strategies.

b) Show that the following pair of mixed strategies is an equilibrium of the mixed game:

$$s_1 = \frac{1}{3}\delta_0 + \frac{1}{6}\delta_{\frac{1}{3}} + \frac{1}{6}\delta_{\frac{2}{3}} + \frac{1}{3}\delta_1$$
$$s_2 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$$

and check that by using such a strategy, a player makes the other one indifferent between all his possible moves.

Problem 11 *Correlated equilibrium*
In the crossing games of example 4, compute all correlated eqilibria. Show that the best symmetric one is a simple "red light".

Problem 12 *more musical chairs*
Consider three variants of example 9 where

- there are two chairs and 3 players
- there are two chairs and 4 players
- there are three chairs and n players, $n \geq 7$

In each case discuss the equilibria in pure strategies, in mixed strategies, and the best symmetric correlated equilibrium.

Problem 13-a *Correlated equilibrium*
We have three players named $1, 2, 3$, each with two strategies labeled A, B. The game is symmetrical, and the payoffs are as follows:

$$\begin{aligned}
(B, B, A) &\rightarrow (2, 2, 0) \\
(A, A, A) \text{ or } (B, B, B) &\rightarrow (1, 1, 1) \\
(B, A, A) &\rightarrow (0, 0, 0)
\end{aligned}$$

a) Find all equilibria in pure strategies, and all equilibria in mixed strategies.
b) Find the symmetrical correlated equilibrium with the largest common payoff.

Problem 13-b *Correlated equilibrium*
This is a symmetric game with four players and each has two pure strategies $s_i \in \{a, b\}$. The payoffs are as follows:

- if all play the same strategy, all get \$10
- if each strategy is played by exactly two players, all get \$16
- if exactly 3 players use the same strategy, these players get \$20, while the remaining player gets \$0.

a) Find all Nash equilibria, if any, in pure strategies. Do not limit yourself to symmetric equilibria.

b) Find all Nash equilibria, if any, in mixed strategies. Do not limit yourself to symmetric equilibria.

c) Find the best (i.e., ensuring the highest expected payoff) symmetric correlated equilibrium.

d) Assume now that if all play the same strategy, all get \$16. Answer questions *b*) and *c*) above.

Problem 14 *a coordination game*

There are q locations equally distributed on the oriented unit circle, $q \geq 3$, and each of the two players chooses one location. The payoff to both players is 1 if they choose the same location, 0 if they choose two different locations that are not adjacent. If the two choices are adjacent, the player who precedes the other (given the orientation of the circle) gets a payoff of 3, the other one gets a payoff of 2.

Show that the game has no pure strategy equilibrium; compute its symmetric equilibrium in mixed strategies and the corresponding payoffs.

Show there is no other equilibrium in mixed strategies.

Construct a correlated equilibrium where total payoff is maximal, namely 2.5 for each player.

Problem 15

Find all equilibria in pure and mixed strategies of the following three person game. Each player has two pure strategies, $C_i = \{x_i, y_i\}$ for all $i = 1, 2, 3$. The payoff is zero to everybody, unless exactly one player i chooses y_i, in which case this player i gets 5, the player before i in the $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ cycle gets 6, and the player after i in this cycle gets 4. Note that the game is not symmetric in the sense of Definition 21 (Chapter 2), yet it is *cyclically* symmetric, i.e., with respect to the cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$.

Compute the (fully) *symmetric* correlated equilibria of the game and compare their payoffs to those of the pure and mixed equilibria.

Problem 16 *Bayesian equilibrium*

a) The strategy sets and information structure is as in Example 10, and the payoffs are

T	$1,2$	$0,0$		T	$0,0$	$3,1$
B	$0,0$	$2,1$		B	$1,3$	$0,0$
t_1	L	R		t_2	L	R

Check that we have two pure strategy equilibria. How many Bayesian equilibria involving mixed strategies?

b) The payoffs are now

T	$1,2$	$0,0$		T	$4,1$	$0,0$
B	$0,0$	$2,1$		B	$0,0$	$2,3$
t_1	L	R		t_2	L	R

Find all Bayesian equilibria.

c) Player 1 chooses a row and his type is known, player 2 chooses a column and his type is t_1 with probability $\frac{2}{3}$, t_2 with probability $\frac{1}{3}$. Payoffs are:

$$
\begin{array}{ccc}
T & 2,0 & 0,2 \\
B & 0,2 & 2,0 \\
t_1 & L & R
\end{array}
\qquad\qquad
\begin{array}{ccc}
T & 0,0 & 2,2 \\
B & 3,3 & 0,0 \\
t_2 & L & R
\end{array}
$$

Find all equilibria in pure strategies and all Bayesian equilibria.

Problem 17

Two opposed armies are poised to seize an island. Each army's general chooses (simultaneously and independently) either to attack or not to attack. In addition, every army is either strong or weak, with equal probability, and the army's type is known to its general (but not to the general of the opposed army). An army captures the island if either it attacks it while its opponent does not attack, or if it attacks while strong, whereas its rival is weak. If two armies of equal strength both attack, neither captures the island.

Payoffs are zero initially; the island is worth 8 if captured; an army incurs a cost of fighting, which is 3 if it is strong and 6 if it is weak. There is no cost of attacking if the rival does not attack, and no cost to not attacking.

Give the normal form of the game, eliminate dominated strategies if any, and compute all Bayesian equilibria.

Problem 18

Mob becomes very strong in fighting on the day he uses drugs, otherwise he is weak. No matter, whether he used drugs or not, Mob is often involved in conflicts of the type described below.

Bob has just insulted Mob in the bar, and Mob must decide whether to fight Bob immediately, or to leave and try to beat Bob after Bob leaves the bar in a couple of hours. If Mob leaves and tries to catch up with Bob later outside, then, if Mob is strong today, he beats Bob and gets utility 10. However, if Mob is weak, Bob beats him and Mob gets -10.

If Mob decides to fight immediately then it is Bob's choice whether to fight or to leave. If Bob leaves, Mob gets utility 5 from humiliating Bob. If they fight in the bar, then on the day Mob is strong he would beat Bob publicly and get utility 20. However, on the day Mob is weak he would loose to Bob publicly and get utility -30.

Mob knows whether he took drugs this day. Bob does not know it, but he was told by the bar owner that Mob uses drugs on average one day out of three.

If Bob is challenged, he gets -10 from leaving, -15 if he fights and looses and 5 if he fights and wins. If the fight is postponed Bob gets -6 from loosing it and 3 from winning.

a) Describe the set of pure strategies for each player, and write the game matrix. Eliminate dominated strategies.

b) Find all Nash equilibria of this game.

Problem 19 *all-pay first price auction*

The game is identical to that in Example 7, except for the fact that the valuation t_i of the object to player i is known only to this agent. Other agents know that t_i is drawn from the uniform probability distribution over $[0, 100]$, and that all draws are stochastically independent.

a) Show that if bidder i observes his type t_i, contemplates the bid y and knows that other bidders all use the same bidding function $x(t)$, bidder i's expected pay-off is

$$t_i \pi\{x(t_j) < y \text{ for all } j \neq i\} - y$$

b) Deduce the unique symmetrical equilibrium bidding function $x(\cdot)$. Compare it to the symmetrical equilibrium of the first price auction.

c) Show that the expected revenue to the seller is the same as in the first price auction (example 11) and in the second price auction. Compare the expected profit of a bidder in these three auctions.

Problem 20 *sealed bid double auction*

In the game of Example 13, consider the following pair of strategies, where α is a number in $[0,300]$:

$$\text{seller } x(a) = \alpha \text{ if } 0 \leq a \leq \alpha; = 300 \text{ if } \alpha < a \leq 300$$

$$\text{buyer } y(b) = 0 \text{ if } 0 \leq b < \alpha; = \alpha \text{ if } \alpha \leq b \leq 300$$

Show that it is a Bayesian equilibrium.

Show that the expected gain of a buyer is $\frac{\alpha(300-\alpha)^2}{2\cdot 300^2}$, and that of a seller is $\frac{\alpha^2(300-\alpha)}{2\cdot 300^2}$. Compute total welfare loss and show that it is never less than 25% (minimum achieved). Compare to the welfare loss of the linear equilibrium found in Example 13.

Problem 21 *alternative trade mechanisms*

As in Example 13, we have a buyer and a seller with IID valuations in $[0, 300]$.

a) Consider the following take it or leave it mechanism: the seller chooses a price $x \in [0, 300]$, which the buyer accepts or not. Compute its unique Bayesian equilibrium, and compare its welfare loss to that found in Example 13, and in Problem 19. also compare the division of the surplus between the two players.

b) Consider the following mechanism. After the seller and buyer independently bid respectively x and y

trade occurs at price $\frac{y}{2}$ if $y \geq 3x$ and $x + y \leq 300$

trade occurs at price $\frac{x}{2} + 150$ if $y \geq \frac{x}{3} + 200$ and $x + y > 300$

no trade occurs, and no money changes hands, in every other case

Show that sincere report of one's valuation ($x(a) = a$ and $y(b) = b$ for all a, b) is a Bayesian equilibrium. Compare the welfare loss of this mechanisms to those found in Example 13 and in Problem 19.

Problem 22 *the lemon problem*

The seller's reservation price t is drawn in $[0, 100]$ with uniform probability. The buyer does not see t. Her reservation price for the object is $\frac{3}{2}x$.

a) Suppose the buyer makes a "take it or leave it" offer which the seller can only accept or reject. Show that the only Bayesian equilibrium of this game has the buyer offering a price of zero, which the seller always refuses.

b) What is the Bayesian equilibrium of the game where the seller makes a "take it or leave it" offer which the buyer can only accept or reject?

Problem 23 *A modified first price auction (50 points)*

There are n potential buyers and one object in the auction. Player i values the object $\$a_i$. Each player submits a bid x_i, then one of the highest bidders gets the object. Denote x^{*1} the highest bid, and x^{*2} the second highest bid. Then the winner of the object pays $\frac{1}{2}(x^{*1} + x^{*2})$. When two or more bidders are tied with the highest bid (so that $x^{*1} = x^{*2}$), we draw as usual one of them with uniform probability to be the winner, who then pays x^{*1}.

a) Assume in this question that all bids x^{*1} and all valuations a_i are in round dollars. Find the undominated strategies in this game, and perform the successive elimination of dominated strategies. Find all Nash equilibria. Which equilibrium (or equilibria) is (are) more likely to emerge when bidders have complete information about the profile of valuations?

b) Assume from now on that all bids x^{*1} and all valuations a_i are real numbers, in other words they are infinitely divisible. Answer the same questions as in a), distinguishing according to the number of players who share the highest valuation.

c) Now the valuations a_i are drawn in $[0, 100]$ independently and according to the same probability distribution with cumulative distribution function F. Find the symmetric Bayesian Nash equilibrium of this game. Recall that if the n random variables Z_i are IID on $[0, 100]$ with c.d.f. F, and Z^{*1}, Z^{*2} are respectively the highest and second highest order statistics, we have

$$E[Z^{*2}|Z^{*1} = t] = \frac{\int_0^t z \, dF^{n-1}(z)}{F^{n-1}(t)}$$

Note: for simplicity you may assume that F is the uniform distribution on $[0, 100]$.