Chapter 3: Nash Equilibrium

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In a general n-person game in strategic form, interests of the players are neither identical nor completely opposed. As in the previous chapter information about other players' preferences and behavior will influence my behavior. The novelty is that this information may sometime be used *cooperatively,* i.e., to our mutual advantage.

We discuss in this chapter the two most important *scenarios* justifying the Nash equilibrium concept as the consequence of rational behavior by the players:

- the *coordinated scenarios* where players know a lot about each other's strategic opportunities (strategy sets) and payoffs (preferences), and use either deductive reasoning or non binding comunication to coordinate their choices of strategies.

- the *decentralized (competitive) scenarios* where mutual information is minimal, to the extent that a player may not even know how many other players are in the game or what their individual preferences look like.

Decentralized scenarios are realistic in games involving a large number of players, each one with a relatively small influence on the overall outcome, so that the "competitive" assumption that each player ignores the influence of his own moves on other players' strategic choices is plausible. Coordination scenarios are more natural in games with a small number of participants.

This chapter is long on examples and short on abstract proofs. The next chapter is just the opposite.

Definition 1 *A game in strategic form is a list $\mathcal{G} = (N, S_i, u_i, i \in N)$, where N is the set of players, S_i is player i's strategy set and u_i is his payoff, a mapping from $S_N = \prod_{i \in N} S_i$ into $\mathbb{R}$, which player i seeks to maximize.*

An important class of games consists of those where the roles of all players are fully interchangeable.

Definition 2 *A game in strategic form $\mathcal{G} = (N, S_i, u_i, i \in N)$ is symmetrical if $S_i = S_j$ for all i, j, and the mapping $s \to u(s)$ from $S^{|N|}$ into $\mathbb{R}^{|N|}$ is symmetrical.*

In a symmetrical game if two players exchange strategies, their payoffs are exchanged and those of other players remain unaffected.

Definition 3 *A Nash equilibrium of the game* $\mathcal{G} = (N, S_i, u_i, i \in N)$ *is a profile of strategies* $s^* \in S_N$ *such that*

$$u_i(s^*) \geq u_i(s_i, s^*_{-i}) \text{ for all } i \text{ and all } s_i \in S_i$$

Note that the above definition uses only the ordinal preferences represented by the utility functions u_i. We use the cardinal representation as payoff (utility) simply for convenience. However when we speak of mixed strategies in the next chapter, the choice of a cardinal utility will matter.

The following inequality provides a useful necessary condition for the existence of at least one Nash equilibrium in a given game $\mathcal{G}$.

Lemma 4 *If* s^* *is a Nash equilibrium of the game* $\mathcal{G} = (N, S_i, u_i, i \in N)$*, we have for all* i

$$u_i(s^*) \geq \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i})$$

Example 1 *duopoly a la Hoteling*
The two competitors sell identical goods at fixed prices p_1, p_2 such that $p_1 < p_2$. The consumers are uniformly spread on $[0, 1]$, each with a unit demand. Firms incur no costs. Firms choose independently where to locate a store on the interval $[0, 1]$, then consumers buy from the cheapest store, taking into account a transportation cost of \$$s$ if s is the distance to the store. Assume $p_2 - p_1 = \frac{1}{4}$. Check that

$$\min_{S_2} \max_{S_1} u_1 = p_1; \min_{S_1} \max_{S_2} u_2 = \frac{p_2}{8}$$

where the $\min_{S_2} \max_{S_1} u_1$ obtains from the copycat strategy $s_1 = s_2$ by player 1, and the $\min_{S_1} \max_{S_2} u_2$ is achieved by $s_1 = \frac{1}{2}$, and $s_2 = 0$ or 1. Observe now that the payoff profile $(p_1, \frac{p_2}{8})$ is not feasible, therefore the game has no Nash equilibrium.

1 Coordinated scenarios

We now consider games in strategic form involving only a few players who use their knowledge about other players strategic options to form expectations about the choices of these players, which in turn influence their own choices. In the simplest version of this analysis, each player knows the entire strategic form of the game, including strategy sets and individual preferences (payoffs). Yet at the time they make their strategic decision, they act independently of one another, and cannot observe the choice of any other player.

The two main interpretations of the Nash equilibrium are then the *self fulfilling prophecy* and the *self enforcing agreement.*

The former is the meta-argument that if a "Book of Rational Conduct" can be written that gives me a strategic advice for every conceivable game in strategic form, this advice must be to play a Nash equilibrium. This is the "deductive' argument in favor of the Nash concept.

The latter assumes the players engage in "pre-play" communication, and reach a non committal agreement on what to play, followed by a complete break up of communication.

Two conceptual difficulties suggest caution when we apply the Nash equilibrium concept in the coordinated context. First a Nash equilibrium may be *inefficient* (Pareto inferior), as illustrated in the celebrated *Prisoner's Dilemna*: section 2.1.1. Then communication between the players drives them to move away from the equilibrium, for the benefit of every participant. Second, many games have multiple Nash equilibria, hence a *selection* problem (section 2.1.2). Under either scenario above, it may be unclear how the players will be able to coordinate on one of them.

On the other hand, we can identify large classes of games in which selecting the Nash outcome by deduction (covert communication) is quite convincing, so that our confidence in the predictive power of the concept remains intact. These are the dominance-solvable games in section 2.1.3, and the games with a dominant strategy equilibrium in section 2.1.4.

1.1 inefficiency of the Nash equilibrium outcomes

Example 2 *Prisonners Dilemna*

Each player chooses a *selfless* strategy C or a *selfish* strategy D. Choosing C brings a benefit a to every *other* player and a cost of b to me. Playing D brings neither benefit nor cost to anyone. It is a dominant strategy to play D if $b > 0$. If furthermore $b < (n-1)a$, the unique Nash equilibrium is Pareto inferior to the unanimously selfless outcome. This equilibrium is especially credible as each player uses a dominant strategy (see 2.1.4 below).

Example 3 *Pigou traffic example*

There are two roads (country, city) to go from A to B and n commuters want to do just that. The country road entails no congestion: no matter how many users travel on it, each incurs a delay of 1. The city road has linear congestion costs: if x commuters use that road, each of them incurs a delay of $\frac{x}{m}$, where we assume $m \leq n$. A Nash equilibrium is an outcome where m, or $m-1$, agents take the city road, and $n-m$, or $n-m+1$, take the country road, and all get a disutility of 1, or $\frac{m-1}{m}$. However total disutility is minimized by sending only $\frac{m}{2}$ commuters on the city road, for a total delay of $n - \frac{m}{4}$, and a Pareto improvement where $\frac{m}{2}$ city commuters are better off, while the rest are indifferent to the change.

Example 4 *the Braess paradox*

There are two roads to go from A to B, and 6 commuters. The upper road goes through C, the lower road goes through D. The 2 roads only meet at A and B. On each of the four legs, AC, CB, AD, DB, the travel time depends upon the number of users x in the following way:

on AC and $DB : 50 + x$, on CB and $AD : 10x$

Every player must choose a road to travel, and seeks to minimize his travel time. The Nash equilibria of the game are all outcomes with 3 users on each road, and they all give the same disutility 83 to each player. Next we add one more link on the road network, directly between C and D, with travel time $10 + x$. In the new Nash equilibrium outcomes, we have two commuters on each of the paths $ACB, ADB, ADCB$, and their disutility is 92. Thus the new road results in a net increase of the congestion!

1.2 the selection problem

When several (perhaps an infinity of) Nash outcomes coexist, and the players' preferences about them do not agree, they will try to force their preferred outcome by means of tactical commitment. Two well known games illustrate the resulting impossibility to predict the outcome of the game.

Example 5 *crossing game (a.k.a. the Battle of the Sexes)*
Each player must stop or go. The payoffs are as follows

stop	$1, 1$	$1-\varepsilon, 2$
go	$2, 1-\varepsilon$	$0, 0$
	stop	*go*

Each player would like to commit to go, so as to force the other to stop. A typical way is unilateral communication (schelling): I am going to pass, I cannot hear you anymore. There is a mixed strategy equilibrium as well, but it has its own problems. See Section 3.3.

Example 6 *Nash demand game*
The two players share a dollar by the following procedure: each write the amounts she demands in a sealed envelope. If the two demands sum to no more than \$1, they are honored. Otherwise nobody gets any money. In this game the equal plit outcome stands out because it is fair, and this will suffice in many cases to achieve coordination. However, a player will take advantage of an opportunity to commit to a high demand. More precisely, the pair s_1, s_2 is a Nash equilibrum if and only if $0 \leq s_1, s_2 \leq 1$ and $s_1 + s_2 = 1$, or $s_1 = s_2 = 1$.

Note that Examples 5 and 6 are symmetric games with (many) asymmetric equilibria.

In both above examples and in the next one the key strategic intuition is that the opportunity to commit to a certain strategy by "burning the bridges" allowing us to play anything else, is the winning move provided one convinces the other player that the bridges are indeed gone.

Definition 5 *Given two functions $t \to a(t)$ and $t \to b(t)$, the corresponding game of timing is as follows. Each one of the two players must choose a time to stop the clock between $t = 0$ and $t = 1$. If player i stops the clock first at time t, his payoff is $u_i = a(t)$, that of player j is $u_j = b(t)$. In case of ties, each gets the payoff $\frac{1}{2}(a(t) + b(t))$.*

An example is the noisy duel of chapter 1, where a increases, b decreases, and they intersect at the optimal stopping/shooting time (here *optimality* refers to the saddle point property for this ordinally zero-sum game). Here is another classic example.

Example 7 *War of attrition*

This is a game of timing where both a and b are continuous and decreasing, $a(t) < b(t)$ for all t, and $b(1) < a(0)$. There are two Nash equilibrium outcomes. Setting t^* as the time at which $a(0) = b(t^*)$, one player commits to t^* or more, and the other concedes by stopping the clock immediately (at $t = 0$).

The selection problem can often be alleviated by further arguments of salience, Pareto dominance, or risk dominance. It is easy to agree on an equilibrium more favorable to everyone: the Pareto dominance argument.

Definition 6 *A coordination game is a game $\mathcal{G} = (N, S_i, u_i, i \in N)$ such that all players have the same payoff function: $u_i(s) = u_j(s)$ for all $i \in N, s \in S_N$.*

If, in a coordination game, there is a single outcome maximizing the common payoff, this Nash equilibrium will be selected without explicit comunication. We have no such luck in a coordination game where several outcomes are optimal, as in Schelling's *rendez-vous game*. Two players living in a big city and unable to communicate directly, must meet tomorrow at noon. If they show up at the same *salient* location (e.g., the Eiffel tower in Paris), they both win a prize, otherwise they get nothing. The problem here is that salience may not be a deterministic criteria.

We illustrate finally the risk dominance argument, in an important model where it conflicts with Pareto dominance.

Example 8 *Coordination failure*

This is an example of a public good provision game by voluntary contributions (example 20, section 2.2.3), where individual contributions enter the common benefit function as perfect complements:

$$u_i(s) = \min_j s_j - C_i(s_i)$$

Examples include the building of dykes or a vaccination program: the safety provided by the dyke is only as good as that of its weakest link. Assume C_i is convex and increasing, with $C_i(0) = 0$ and $C_i'(0) < 1$, so that each player has a stand alone optimal provision level s_i^* maximizing $z - C_i(z)$. Then the Nash equilibria are the outcomes where $s_i = \lambda$ for all i, and $0 \leq \lambda \leq \min_i s_i^*$. They are Pareto ranked: the higher λ, the better for everyone. However the higher λ, the more risky the equilibrium: if other players may make an error and fail to send their contribution, it is prudent not to send anything ($\max_{s_i} \min_{s_{-i}} u_i(s) = 0$ is achieved with $s_i = 0$). Even if the probability of an error is very small, a reinforcement effect will amplify the risk till the point where only the null (prudent) equilibrium is sustainable.

1.3 dominance solvable games

Eliminating dominated strategies is the central coordination device performed by independent deductions of players mutually informed about the payoff functions. We repeat a definition already given for two-person zero-sum games (Definition 13).

Definition 7 *In the game $\mathcal{G} = (N, S_i, u_i, i \in N)$, we say that player i's strategy s_i is weakly dominated by his strategy s'_i (or simply dominated) if*

$$\begin{aligned} u_i(s_i, s_{-i}) &\leq u_i(s'_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i} \\ u_i(s_i, s_{-i}) &< u_i(s'_i, s_{-i}) \text{ for some } s_{-i} \in S_{-i} \end{aligned}$$

We say that strategy s_i is strictly dominated by s'_i if

$$u_i(s_i, s_{-i}) < u_i(s'_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

Given a subset of strategies $T_i \subset S_i$ we write $\mathcal{WU}_i(T_N)$ (resp. $\mathcal{U}_i(T_N)$) for the set of player i's strategies in the restricted game $(N, T_i, u_i, i \in N)$ that are not dominated (resp. not strictly dominated).

Definition 8 *We say that the game $\mathcal{G}$ is dominance-solvable (resp. strictly dominance-solvable) if the sequence defined inductively by*

$$^wS_i^0 = S_i;^w S_i^{t+1} = \mathcal{WU}_i(^wS_N^t) \text{ (resp. } S_i^0 = S_i; S_i^{t+1} = \mathcal{U}_i(S_N^t)) \text{ for all } i \text{ and } t = 1, 2, \cdots$$

and called the successive elimination of dominated (resp. strictly dominated) strategies, converges to a single outcome s^*:

$$\cap_{t=1}^{\infty} {}^wS_N^t = \{s^*\} \text{ (resp. } \cap_{t=1}^{\infty} S_N^t = \{s^*\})$$

If the strategy sets are finite, or compact with continuous payoff functions, the set of undominated strategies $\mathcal{U}_i(S_N)$ is non empty and closed, therefore the sequence S_N^t is well defined. On the other hand, the (smaller) set of weakly undominated strategies $\mathcal{WU}_i(S_N)$ is non empty but it may not be closed. Therefore the existence of the sequence ${}^wS_N^t$ is not always guaranteed, because in the second round we lose compactness. The issue does not arise if the strategy sets are finite.

Despite their close similarities, the two types of elimination, of dominated or of strictly dominated strategies, differ in other important ways. The latter never throws away a Nash equilibrium outcome, and so it is not a selection tool, rather a way to identify games with a unique Nash equilibrium. The former, on the other hand, is a genuine selection tool, but one that must be handled with care.

Proposition 9 *For any T the set $\cap_{t=1}^{T} S_N^t$ contains all Nash equilibria of the game. If $\cap_{t=1}^{\infty} S_N^t = \{s^*\}$, then s^* is the single Nash equilibrium outcome of the original game.*

If the strategy sets are finite and $\cap_{t=1}^{\infty} {}^wS_N^t = \{s^*\}$, then s^* is a Nash equilibrium of the original game.

The successive elimination of strictly dominated strategies is very robust in the sense that it never loses equilibria , whereas the successive elimination of weakly dominated strategies may lose some, or even all Nash equilibria of the original game (in the latter case, the game reduced to $\cap_{t=1}^{T}{}^{w}S_N^t$ contains no equilibrium either). Here are two examples

$$\begin{bmatrix} 1,0 & 2,0 & 1,5 \\ 6,2 & 3,7 & 0,5 \\ 3,1 & 2,3 & 4,3 \end{bmatrix}$$

where the elimination of weakly d.s. picks one of the two equilibria, and

$$\begin{bmatrix} 1,3 & 2,0 & 3,1 \\ 0,2 & 2,2 & 0,2 \\ 3,1 & 2,0 & 1,3 \end{bmatrix}$$

where the algorithm throws out the unique Nash equilibrium!

Another difference between the two successive elimination algorithms, based on strict or weak domination, is their robustness with respect to partial elimination. Suppose that at each stage we only drop strictly dominated strategies, i.e., we construct a sequence R_i^t such that $R_i^{t+1} \subseteq \mathcal{U}_i(R_N^t)$ for all i and t. Then it is easy to check that the limit set $\cap_{t=1}^{\infty} R_N^t$ is unaffected, provided we do eliminate some strategies at each round (see Problem 10). On the other hand when we only drop some weakly dominated strategies at each stage, the result of the algorithm may well depend on which ones we drop. Here is an example:

$$\begin{bmatrix} 2,3 & 2,3 \\ 3,2 & 1,2 \\ 1,1 & 0,0 \\ 0,0 & 1,1 \end{bmatrix}$$

Depending on which strategy player 1 eliminates first, we end up at the $(3,2)$ or the $(2,3)$ equilibrium.

The bottom line is that the successive elimination of strictly dominated strategies can be performed without thinking twice, while we must be cautious in performing the successive elimination of strictly dominated strategies, that can lead to paradoxical results. We use several classic examples to reinforce this point.

Example 9 *Guessing game*
Each one of the n players chooses an integer s_i between 1 and 1000. Compute the average response

$$\overline{s} = \frac{1}{n}\sum_i s_i$$

Each player receives a prize that strictly decreases with the distance of its own strategy s_i to $\frac{2}{3}\overline{s}$

$$u_i(s) = -f(|s_i - \frac{2}{3}\overline{s}|)$$

This game is strictly dominance solvable and

$$\cap_{t=1}^{\infty} S_N^t = \{(1, \cdots, 1)\}$$

Observe that for any $t = 0, 1, \cdots$, if $S_i^t \subseteq \{1, \cdots, p\}$ for some integer p, then $S_i^{t+1} \subseteq \{1, \cdots, \lceil \frac{2}{3} p \rceil\}$. To prove this claim we check that player i's strategy $s_i^* = \lceil \frac{2}{3} p \rceil$ strictly dominates any strategy s_i such that $s_i \geq s_i^* + 1$. Assume player i uses s_i^* and denote by $\widetilde{s}$ the average strategy of players other than i, so that $\overline{s} = \frac{1}{n} s_i^* + \frac{n-1}{n} \widetilde{s}$. Simple computations give

$$\widetilde{s} \leq p \Rightarrow s_i^* \geq \frac{2}{3} \overline{s} \text{ and } s_i^* - \frac{2}{3} \overline{s} < s_i - \frac{2}{3} (\frac{1}{n} s_i + \frac{n-1}{n} \widetilde{s})$$

so s_i^* is strictly closer to $\widetilde{s}$ than s_i. We can now apply the upper bound on S_i^{t+1} repeatedly:

$S_i^1 \subseteq \{1, \cdots, 667\}, S_i^2 \subseteq \{1, \cdots, 445\}, \cdots, S_i^8 \subseteq \{1, \cdots, 40\}, \cdots, S_i^{16} \subseteq \{1, 2\}$. Finally if the game is reduced to the strategies 1 and 2 for everyone, check that strategy 2 is at least $\frac{2}{3}$ away from $\frac{2}{3} \overline{s}$, while strategy 1 is at most $\frac{1}{3}$ away from $\frac{2}{3} \overline{s}$.

The guessing game has been widely tested in the lab, where the participants' limited strategic sophistication lead them to perform only a couple (typically two or three) of rounds of elimination. When playing the guessing game with inexperienced opponents, it is therefore a good idea to choose a number between $(\frac{2}{3})^2 50 \simeq 19$ and $(\frac{2}{3})^3 50 \simeq 12$.

Example 10 *Cournot duopoly*

Firm i produces s_i units of output, at a unit cost of c_i. The price at which the total supply $s_1 + s_2$ clears is $[A - (s_1 + s_2)]_+$. Hence the profit functions:

$$u_i = [A - (s_1 + s_2)]_+ s_i - c_i s_i \text{ for } i = 1, 2$$

This game is strictly dominance-solvable.

In our next example, weak dominance solvability leads to a mildly paradoxical result.

Example 11 *The chair's paradox*

Three voters choose one of three candidates a, b, c. The rule is plurality with the Chair, player 1, breaking ties. Hence each player i chooses from the set $S_i = \{a, b, c\}$, and the elected candidate for the profile of votes s is

$$s_2 \text{ if } s_2 = s_3; \text{ or } s_1 \text{ if } s_2 \neq s_3$$

Note that the Chair has a dominant strategy (Definition 25 below) to vote for her top choice. The two other players can only eliminate the vote for their bottom candidate as (weakly) dominated.

Assume that the preferences of the voters exhibit the cyclical pattern known as the *Condorcet paradox*, namely

$$u_1(c) < u_1(b) < u_1(a)$$

$$u_2(b) < u_2(a) < u_2(c)$$
$$u_3(a) < u_3(c) < u_3(b)$$

Writing this game in strategic form reveals that after the successive elimination of dominated strategies, the single outcome $s = (a, c, c)$ remains. This is the most plausible Nash equilibrium outcome when players know all preferences. The paradox is that the chair's tie-breaking privilege result in the election of her worst outcome! There are other equilibria; two examples are: everyone votes for a, or everyone for b.

In spite of the shortcomings detailed above, in many important economic games, a couple of rounds of elimination of weakly dominated strategiesf may well be enough to select a unique Nash equilibrium, even though the elimination algorithm is stopped and the initial game is not weakly dominance solvable.

Example 12 *First price auction*
The sealed bid first price auction is strategically equivalent to the Dutch descending auction.An object is auctioned between n bidders who each submit a sealed bid s_i. Bids are in round dollars (so $S_i = \mathbb{N}$). The highest bidder gets the object and pays his bid. In case of a tie, a winner is selected at random with uniform probability among the highest bidders. Assume that the valuations of (willingness to pay for) the object are also integers u_i and that

$$u_1 > u_i \text{ for all } i \geq 2$$

At a Nash equilibrium of this game, the object is awarded to player 1 at a price anywhere between $u_1 - 1$ and u_2, and there is another bid just below player 1's winning bid. However after two rounds of elimination we find a game where the only Nash equilibrium has player 1 paying u_2 for the object while one of the players i, $i \geq 2$, such that $u_i = \max_{j \neq 1} u_j$ bids $u_i - 1$. Thus player 1 exploits his informational advantage to the full.

Example 13 *Steinhaus cake division method*
The referee runs a knife from the left end of the cake to its right end. Each one of the two players can stop the knife at any moment. Whoever stops the knife first gets the left piece,the other player gets the right piece.
If both players have identical preferences over the various pieces of the cake, this is a game of timing structurally equivalent to the noisy duel, and its unique Nash equilibrium is that they both stop the knife at the time t^* when they are indifferent between the two pieces.
When preferences differ, this is a variant of a game of timing. Call t_i^* the time when player i is indifferent between the two pieces, and assume $t_1^* < t_2^*$. The Nash equilibrium outcomes are those where player 1 stops the knife between t_1^* and t_2^* while player 2 is just about to stop it herself: player 1 gets the left piece (worth more than the right piece to him) and player 2 gets the right piece (worth more to her than the left piece). However after two rounds of elimination of weakly dominated strategies, we are left with $S_1^2 = [t_2^* - \varepsilon, 1], S_2^2 = [t_2^*, 1]$ (where our notation is loose; for a precise statement, it is easier to give a discrete version of the model). Although the elimination process stops there, the outcome of the

remaining game is not in doubt: $s_1^* = t_2^* - \varepsilon, s_2^* = t_2^*$. Indeed the remaining game is *inessential* (see Problem 29, question a).

1.4 dominant strategy equilibrium

One instance where the successive elimination of weakly dominated strategies is convincing is when each player has a dominant strategy. Put differently, the following is a compelling equilibrium selection.

Definition 10 *In the game $\mathcal{G} = (N, S_i, u_i, i \in N)$, we say that player i's strategy s_i^* is dominant if*

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}, \text{ all } s_i \in S_i$$

We say that s^* is a dominant strategy equilibrium if for each player i, s_i^* is a dominant strategy.

There is a huge difference in the interpretation of a game where dominance solvability (whether in the strict or weak form) identifies a Nash equilibrium, versus one where a dominant strategy equilibrium exists.

The former requires complete information about mutual preferences and more: I know your preferences, I know that you know that I know your preferences, etc..You know my preferences, I know that you know my preferences, you know that I know ...

In the latter, all a player has to know are the strategy sets of other players; their preferences or their actual strategic choices do not matter at all to pick his dominant strategy. Strategic choices are truly decentralized. Information about other players' payoffs or moves is worthless, as long as our player is unable to influence their choices (no direct communication channel allows to convey a threat of the kind "if you do this I will do that", or this threat is not enforceable).

The Prisoner's Dilemna (Example 1) is the most famous instance of a game with a dominant strategy equilibrium.

Dominant strategy equilibria are rare because the strategic interaction is often more complex. However they are so appealingly simple that when we design a procedure to allocate resources, elect one of the candidates to a job, or divide costs, we would like the corresponding strategic game to have a dominant strategy equilibrium as often as possible. In this way we are better able to predict the behavior of our participants. The two most important examples of such *strategy-proof* allocation mechanisms follow. In both cases the game has a (weakly but not strictly) dominant strategy equilibrium for all preference profiles, and the corresponding outcome is efficient (Pareto optimal).

Example 14 *Vickrey's second price auction*
An object is auctioned between n bidders who each submit a sealed bid s_i. Bids are in round dollars (so $S_i = \mathbb{N}$). The highest bidder gets the object and pays *the second highest bid.* In case of a tie, a winner is selected at random with uniform probability among the highest bidders (and pays the highest bid). If player i's

valuation of the object is u_i, it is a dominant strategy to bid "sincerely", i.e., $s_i^* = u_i$. The corresponding outcome is the same as in the Nash equilibrium of the first price auction that we selected by dominance-solvability in example 12. But to justify that outcome we needed to assume complete information, in particular the highest valuation player must know precisely the second highest valuation. By contrast in the Vickrey auction, each player knows what bid to slip in the envelope, whether or not she has any information about other players' valuations, or even their number.

Note that in the second price auction game, there is a distressing variety of other Nash equilibrium outcomes. In particular any player, even the one with the lowest valuation of all, receives the object in some equilibrium. It is easy to check that for any player i and for any price p, $0 < p < a_i$ there is a Nash equilibrium where player i gets the object and pays p.

Example 15 *voting under single-peaked preferences*
The n players vote to choose an outcome x in $[0, 1]$. Preferences of player i over the outcomes are single-peaked with the peak at v_i: they are strictly increasing on $[0, v_i]$ and strictly decreasing on $[v_i, 1]$. Assume for simplicity n is odd. Each player submits a ballot $s_i \in [0, 1]$, and the *median* outcome among $s_1, \cdots, s_n$ is elected: this is the number $x = s_{i^*}$ such that more than half of the ballots are no less than x, and more than half of the ballots are no more than x.

It is a dominant strategy to bid "sincerely", i.e., $s_i^* = v_i$. Again, any outcome x in $[0, 1]$ results from a Nash equilibrium, so the latter concept has no predictive power at all in this game.

2 Decentralized behavior and dynamic stability

In this section we interpret a Nash equilibrium as the resting point of a dynamical system. The players behave in a simple myopic fashion, and learn about the game by exploring their strategic options over time. Their behavior is compatible with total ignorance about the existence and characteristics of other players, and what their behavior could be.

Think of Adam Smith's *invisible hand* paradigm: the price signal I receive from the market looks to me as an exogenous parameter on which my own behavior has no effect. I do not know how many other participants are involved in the market, and what they could be doing. I simply react to the price by maximizing my utility, without making assumptions about its origin.

The analog of the *competitive behavior* in the context of strategic games is the *best reply behavior*. Take the profile of strategies s_{-i} chosen by other players as an exogeneous parameter, then pick a strategy s_i maximizing your own utility u_i, under the assumption that this choice will not affect the parameter s_{-i}.

The deep insight of the invisible hand paradigm is that decentralized price taking behavior will result in an efficient allocation of resources (a Pareto efficient outcome of the economy). This holds true under some specific microeconomic assumptions in the Arrow-Debreu model, and consists of two statements.

First the invisible hand behavior will converge to a competitive equilibrium; second, this equilibrium is efficient. (The second statement is much more robust than the first).

In the much more general strategic game model, the limit points of the best reply behavior are the Nash equilibrium outcomes. Both statements, the best reply behavior converges, the limit point is an efficient outcome, are problematic. The examples below show that the best reply behavior may not converge at all If it converges, the limit Nash equilibrium outcome may well be inefficient (as we saw in section 2.1). Decentralized behavior may diverge, or it may converge toward a socially suboptimal outcome.

2.1 Stable and unstable equilibria

Definition 11 *Given the game in strategic form $\mathcal{G} = (N, S_i, u_i, i \in N)$, the best-reply correspondence of player i is the (possibly multivalued) mapping br_i from $S_{-i} = \prod_{j \in N \setminus \{i\}} S_j$ into S_i defined as follows*

$$s_i \in br_i(s_{-i}) \Leftrightarrow u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \text{ for all } s'_i \in S_i$$

Definition 12 *We say that the sequence $s^t \in S_N, t = 0, 1, 2, \cdots,$ is a* ***best reply dynamics*** *if for all $t \geq 1$ and all i, we have*

$$s_i^t \in \{s_i^{t-1}\} \cup br_i(s_{-i}^{t-1}) \text{ for all } t \geq 1$$

$$\text{and } s_i^t \in br_i(s_{-i}^{t-1}) \text{ for infinitely many values of } t$$

We say that s^t is a ***sequential best reply dynamics*** *if in addition at each step at most one player is changing her strategy.*

The best reply dynamics is very general, in that it does not require the successive adjustments of the players to be synchronized. If all players use a best reply at all times, we speak of *myopic adjustment;* in a sequential best reply dynamics, players take turn to adjust. For instance with two players the latter dynamics is:

$$\text{if } t \text{ is even: } s_1^t \in br_i(s_2^{t-1}), s_2^t = s_2^{t-1}$$

$$\text{if } t \text{ is odd: } s_2^t \in br_i(s_1^{t-1}), s_1^t = s_1^{t-1}$$

But the definition allows much more complicated dynamics, where the timing of best reply adjustments varies accross players. An important requirement is that at any date t, every player will be using his best reply adjustment some time in the future.

The first observation is an elementary result.

Proposition 13 *Assume the strategy sets S_i of each player are compact and the payoff functions u_i are continuous (this is true in particular if the sets S_i are finite). If the best reply dynamics $(s^t)_{t \in \mathbb{N}}$ converges to $s^* \in S_N$, then s^* is a Nash equilibrium.*

Proof. Pick any $\varepsilon > 0$. As u_i is uniformly continuous on S_N, there exists T such that

$$\text{for all } i, j \in N \text{ and } t \geq T\text{: } |u_i(s_j^t, s_{-j}) - u_i(s_j^*, s_{-j})| \leq \frac{\varepsilon}{n} \text{ for all } s_{-j} \in S_{-j}$$

Fix an agent i. By definition of the b.r. dynamics, there is a date $t \geq T$ such that $s_i^{t+1} \in br_i(s_{-i}^t)$. This implies for any $s_i \in S_i$

$$u_i(s^*) + \varepsilon \geq u_i(s_i^{t+1}, s_{-i}^t) \geq u_i(s_i, s_{-i}^t) \geq u_i(s_i, s_{-i}^*) - \frac{n-1}{n}\varepsilon$$

where the left and right inequalities follow by repeated application of uniform continuity. Letting ε go to zero ends the proof. ■

Observe that a limit point s^* of the best reply dynamics $(s^t)_{t \in \mathbb{N}}$ may *not be* a Nash equilibrium! Indeed if strategy sets are compact, any best reply dynamics has some limit points, while the initial game may not have any Nash equilibrium, and in that case Proposition 29 says that no best reply dynamics ever converges. The second game in Example 16 below is a case in point; see also Example 19.

Definition 14 *We call a Nash equilibrium s* ***strongly stable*** *if any best reply dynamics (starting form any initial profile of strategies in S_N) converges to s. Such an equilibrium must be the unique equilibrium.*

We call a Nash equilibrium ***sequentially stable*** *if any sequential best reply dynamics (starting form any initial profile of strategies in S_N) converges to it. Such an equilibrium must be the unique equilibrium.*

We give a series of examples illustrating these definitions.

Example 16: *Two-person zero sum games*

Here a Nash equilibrium is precisely a saddle point. In the following game, a saddle point exists and is strongly stable

$$\begin{bmatrix} 4 & 3 & 5 \\ 5 & 2 & 0 \\ 2 & 1 & 6 \end{bmatrix}$$

Check that 3 is the value of the game. To check stability check that from the entry with payoff 1, any b.r. dynamics converges to the saddle point; then the same is true from the entry with payoff 6; then also from the entry with payoff 0, and so on.

In the next game, a saddle point exists but is not even sequentially stable:

$$\begin{bmatrix} 4 & 1 & 0 \\ 3 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$$

Starting from (top,left), say, we cycle on the four corners of the matrix, each one of them a limit point of the sequential b.r. dynamics, but we never reach the saddle point (middle,middle).

Stability in finite a (not necessarily zero-sum) two person game (S_1, S_2, u_1, u_2) is easy to analyze. Define $f = br_2 \circ br_1$ the composition of the two best reply correspondences. A fixed point of f is $s_2 \in S_2$ such that $s_2 \in f(s_2)$, and a cycle of length T is a sequence of *distinct* elements $s_2^t, t = 1, \cdots, T$ such that $s_2^{t+1} \in f(s_2^t)$ for all $t = 1, \cdots, T-1$, and $s_2^1 \in f(s_2^T)$.

Proposition 15 *The Nash equilibrium s^* of the finite game (S_1, S_2, u_1, u_2) is strongly stable if and only if it is sequentially stable. This happens if and only if f has a unique fixed point and no cycle of length 2 or more.*

Proof. If the game is sequentially stable, a sequence s_2^t with an arbitrary starting point s_2 and such that $s_2^{t+1} \in f(s_2^t)$, converges to the same limit s_2^*, and the corresponding sequence s_1^t also has a unique limit s_1^*. Thus (s_1^*, s_2^*) is the unique Nash equilibrium outcome and s_2^* the unique fixed point of f. To check strong stability, consider any best reply dynamics s^t. At some $t \geq 1$, $s_1^{t+1} \in br_1(s_2^t)$, so s^t reaches the set $br_1(S_2) \times S_2$, and never leaves it thereafter. at some $t' > t$, $s_2^{t'+1} \in br_2(s_1^{t'})$, so the sequence s^t reaches the set $br_1(S_2) \times f(S_2)$, never to leave it. Repeating the argument, we see that the sequence reaches $br_1 \circ f(S_2) \times f(S_2)$, then $br_1 \circ f(S_2) \times f^2(S_2)$, and so on, which ensures its convergence to (s_1^*, s_2^*).

The easy proof of the second statement is omitted. ■

Example 17 *price cycles in the Cournot oligopoly*
The demand function and its inverse are

$$D(p) = (a - bp)_+ \Leftrightarrow D^{-1}(q) = \frac{(a-q)_+}{b}$$

Firm i incurs the cost $C_i(q_i) = \frac{q_i^2}{2c_i}$ therefore its competitive supply given the price p is $O_i(p) = c_i p$, and total supply is $O(p) = (\sum_N c_i)p$. Assume there are many agents, each one small w.r.t. the total market size (i.e., each c_i is small w.r.t. $\sum_N c_j$), so that the competitive price-taking behavior is a good approximation of the best reply behavior. Strategies here are the quantities q_i produced by the firms, and utilities are

$$u_i(q) = D^{-1}(\sum_N q_j) q_i - C_i(q_i)$$

The equilibrium is unique, at the intersection of the O and D curves: the myopic adjustment dynamics follows $q \to D^{-1}(q) = p \to O(p) = q' \to D^{-1}(q') \to \cdots$. Set $c = \sum_N c_j$; if $\frac{b}{c} > 1$ the equilibrium is strongly stable; if $\frac{b}{c} < 1$ it is sequentially but not strongly stable.

Example 18: *Schelling's model of binary choices*
Each player has a binary choice, $S_i = \{0, 1\}$, and the game is symmetrical, therefore it is represented by two functions $a(.), b(.)$ as follows

$$\begin{aligned} u_i(s) &= a(\frac{1}{n}\sum_N s_i) \text{ if } s_i = 1 \\ &= b(\frac{1}{n}\sum_N s_i) \text{ if } s_i = 0 \end{aligned}$$

Assuming a large number of agents, we can draw a, b as continuous functions and check that the Nash equilibrium outcomes are at the intersections of the 2 graphs, at $s = (0, \cdots, 0)$ if $a(0) \leq b(0)$, and at $s = (1, \cdots, 1)$ if $a(1) \geq b(1)$.

Whether a cust b from above or below makes a big difference in the stability of the corresponding equilibrium outcome.

Example 18a: vaccination Strategy 1 is to take the vaccine, strategy 0 to avoid it. Both a and b are strictly increasing: the risk of catching the disease diminishes as more people around us vaccinate. If $\frac{1}{n}\sum_N s_i$ is very small, $a > b$, as the risk of catching the disease is much larger than the risk of complications from the vaccine; this inequality is reversed when $\frac{1}{n}\sum_N s_i$ is close to 1. So the intersection of the two curves is the sequentially stable, but not strongly stable, equilibrium outcome[1].

Example 18b: traffic Each player chooses to use the bus ($s_i = 1$) or his own car ($s_i = 0$); for a given congestion level $\frac{1}{n}\sum_N s_i$, traffic is equally slow in either vehicle, but more comfortable in the car, so $a(t) < b(t)$ for all t; however a and b both increase in t, as more people riding the bus decreases congestion. If $a(1) > b(0)$ the equilibrium in dominant strategies $s_i = 0$ for all i, is Pareto inferior. It is strongly stable.

Example 18c Now a and b intersect only once, and a cuts b from below. We have three equilibrium outcomes, at $t = 0, 1$ and at the intersection of a and b. The latter is unstable, and the former two are stable in a "local" sense[2].

2.2 strictly dominance-solvable games

We saw in section 2.1.3 that in such games the Nash equilibrium exists and is unique. We can say more.

Proposition 16 *If the game $\mathcal{G} = (N, S_i, u_i, i \in N)$ is strictly dominance solvable, its unique Nash equilibrium $\cap_{t=1}^{\infty} S_N^t = \{s^*\}$ is strongly stable.*

The unique equilibrium obtains both as the result of the (timeless) deductive process of successive elimination of strategies by fully informed players, and also as the limit of any best reply dynamics by players with very limited knowledge of their environment who naively "best reply" to the observed behavior of the other players (unaware of those players' preferences).

See section 2.1.3 for examples.

[1] Note that this is a statement *in utilities*, as the Nash equilibrium property only determines the number of players using each strategy, but not their identity. Yet all equilibrium outcomes yield the same utility profile, which allows us to state those stability properties.

[2] We measure the deviation from an equilibrium by the number of agents who are not playing the equilibrium strategy. We say that a a Nash equilibrium s^* is *locally stable in population* if there exists a number $\lambda, 0 < \lambda < 1$, such that if no more than a fraction λ of the population deviates from it, any sequential dynamics converges back to s^*.

2.3 potential games

Potential games generalize the pure coordination games (Definition 21) where all players have the same payoff functions. As shown in the following example, strong stability is problematic in a coordination game but sequential stability is not.

Example 19 *a simple coordination game*
The game is symmetrical and the common strategy space is $S_i = [0,1]$; the payoffs are identical for all n players

$$u_i(s) = g(\sum_{i=1}^{n} s_i)$$

where g is a continuous function on $[0,n]$.
Suppose that g has a unique maximum z^* and no other local maxima (g is single-peaked). All s such that $\sum_{i=1}^{n} s_i = z^*$ are Nash equilibria, therefore none is strongly stable. The single exceptions are $z^* = 0$ or 1, because then the Nash equilibrium is unique, and strongly stable. For z^* such that $0 < z^* < 1$, the *game* is sequentially stable *in utilities* (Example 18), because along any sequential best reply dynamics, the common utility increases and converges to $g(z^*)$. However, even in the restricted sense of convergence in utilities, the game is not strongly stable, because, for instance, simultaneous best reply sequences cycle around z^* without reaching it.

Definition 17 *A game in strategic form $\mathcal{G} = (N, S_i, u_i, i \in N)$ is a potential game if there exists a real valued function P defined on S_N such that for all i and $s_{-i} \in S_{-i}$ we have*

$$u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) = P(s_i, s_{-i}) - P(s'_i, s_{-i}) \text{ for all } s_i, s'_i \in S_i$$

or equivalently there exists P and for all i a real valued function h_i defined on $S_{N \diagdown \{i\}}$ such that

$$u_i(s) = P(s) + h_i(s_{-i}) \text{ for all } s \in S_N$$

The original game $\mathcal{G} = (N, S_i, u_i, i \in N)$, and the game $\mathcal{P} = (N, S_i, P, i \in N)$ with the same strategy sets as $\mathcal{G}$ and identical payoffs P for all players, have the same best reply correspondences therefore the same Nash equilibria. Call s^* a *coordinate-wise maximum* of P if for all i, $s_i \to P(s_i, s^*_{-i})$ reaches its maximum at s^*_i. Clearly s is a Nash equilibrium (of $\mathcal{G}$ and $\mathcal{P}$) if and only if it is a coordinate-wise maximum of P.

If P reaches its global maximum on S_N at s, this outcome is a Nash equilibrium of $\mathcal{P}$ and therefore of $\mathcal{G}$. Thus potential games with continuous payoff functions and compact strategy sets always have at least a Nash equilibrium. Moreover, these equilibria have appealing stability properties.

Proposition 18 *Let $\mathcal{G} = (N, S_i, u_i, i \in N)$ be a potential game where the sets S_i are compact and the payoff functions u_i are continuous. If the best reply function of every player is single valued and continuous, and the Nash equilibrium is unique, the game $\mathcal{G}$ is sequentially stable.*

Proof. (sketch) For any sequential b.r. dynamics s^t if $s^t \neq s^{t+1}$, we have $P(s^t) < P(s^{t+1})$, because the best reply functions are single valued. If the sequence s^t has more than one limit point, one constructas a cycle of the sequential best reply dynamics, which contradicts the fact that P strictly increases along such dynamics. Thus s^t converges, and by continuity of P it must be a coordinate-wise maximum of P, namely a Nash equilibrium. ■

Example 20 *public good provision by voluntary contributions*
Each player i contributes an amount of input s_i toward the production of a public good, at a cost $C_i(s_i)$. The resulting level of public good is $B(\sum_i s_i) = B(s_N)$. Hence the payoff functions

$$u_i = B(s_N) - C_i(s_i) \text{ for } i = 1, \cdots, n$$

The potential function is

$$P(s) = B(s_N) - \sum_i C_i(s_i)$$

therefore existence of a Nash equilibrium is guaranteed if B, C_i are continuous and the potential is bounded over $\mathbb{R}_+^N$.

Remark The public good provision model is a simple and compelling argument in favor of centralized control of the production of pure public goods. To see that in equilibrium the level of production is grossly inefficient, assume for simplicity identical cost functions $C_i(s_i) = \frac{1}{2}s_i^2$ and $B(z) = z$. The unique Nash equilibrium is $s_i^* = 1$ for all i, yielding total utility

$$\sum_i u_i(s^*) = nB(s_N^*) - \sum_i C_i(s_i^*) = n^2 - \frac{n}{2}$$

whereas the outcome maximizing total utility is $\widetilde{s}_i = n$, bringing $\sum_i u_i(\widetilde{s}) = \frac{n^3}{2}$, so each individual equilibrium utility is less than $\frac{2}{n}$ of its "utilitarian" level.

The much more general version of Example 20 where the common benefit is an arbitrary function $B(s) = B(s_1, \cdots, s_n)$, remains a potential game for $P = B - \sum_i C_i$, therefore existence of a Nash equilibrium is still guaranteed. See Example 8 and Problem 7 for two alternative choices of B, respectively $B(s) = \min s_i$ and $B(s) = \max s_i$.

Example 21 *congestion games*

These games generalize both Pigou's model (Example 3) and Schelling's model (Example 18). Each player i chooses from the same strategy set and her payoff only depends upon the number of other players making the same choice. Examples include choosing a travel path between a source and a sink when delay is the only consideration, choosing a club for the evening if crowding is the only criteria, and so on.

$S_i = S$ for all i; $u_i(s) = f_{s_i}(n_{s_i}(s))$ where $n_x(s) = |\{j \in N | s_j = x\}|$ and f_x is arbitrary. If f is decreasing, we have a negative congestion externality, as in traffic examples. If f is increasing we have the opposite effect where we want more players to choose the same strategy as our own, as in the club example.

In the latter we can also think of f as single-peaked (some crowding is good, up to a point).

Here the potential function is

$$P(s) = \sum_{x \in S} \sum_{m=1}^{n_x(s)} f_x(m)$$

See Problems 16 to 19 for illustrations and variants.

3 Problems on chapter 3

Problem 1
In Schelling's model (example 18) find the Nash equilibrium outcomes and analyze their stability in the following cases:

a) $a(t) = 8t(1-t); b(t) = t$
b) $a(t) = 8t(1-t); b(t) = 1-t$
c) $a(t) = 8t(1-t); b(t) = \frac{1}{2}$

Problem 2 *Games of timing (Definition 20)*
a) We have two players, a and b both increase, are continuous, and a intersects b from below only once. Perform the successive elimination of (weakly and strictly) dominated strategies, and find all Nash equilibria. Can they be Pareto improved?
b) We extend the war of attrition (example 7) to n players. If player i stops the clock first at time t, his payoff is $u_i = a(t)$, that of all other players is $u_j = b(t)$. Both a and b are continuous and decreasing, $a(t) < b(t)$ for all t, and $b(1) < a(0)$. Answer the same questions as in a).
c) We have n players as in question b), but this time a increases, b decreases, and they intersect.

Problem 3 *Example 13 continued*
The interval $[0, 1]$ is a nonhomogeneous cake to be divided between two players. The utility of player 1 for a share $A \subset [0, 1]$ is $v_1(A) = \int_A \left(\frac{3}{2} - x\right) dx$. The utility of player 2 for a share $B \subset [0, 1]$ is $v_2(B) = \int_B \left(\frac{1}{2} + x\right) dx$. When time runs from $t = 0$ to $t = 1$, a knife is moved at the speed 1 from $x = 0$ to $x = 1$. Each player can stop it at any time. If the knife is stopped at time t by player i, this player gets the share $[0, t]$, while the other player gets the share $[t, 1]$.

Analyze the game as in Example 13. What strategic advice would you give to each player? Distinguish the two cases where this player knows his opponent's utility and that where she does not.

Problem 4
One hundred people live in the village, of whom 51 support the conservative candidate and 49 support the liberal candidate. A villager gets utility +9 if her candidate wins, -11 if her candidate looses, and 0 if they are tied. In addition, she gets a disutility of -1 for actually voting, but no disutility for staying home (so if her canddate wins and she voted, net utility is 10, etc..).

a) Why it is not Nash equilibrium for everybody to vote?
b) Why it is not Nash equilibrium for nobody to vote?
c) Find a Nash equilibrium where all conservatives use the same strategy, and all liberals use the same strategy.
d) What can you say about other possible Nash equilibria of this game?

Problem 5 *third price auction*
We have n bidders, $n \geq 3$, and bidder i's valuation of the object is u_i. Bids are independent and simultaneous. The object is awarded to the highest bidder at the third highest price. Ties are resolved just like in the Vickrey auction, with the winner still paying the third highest price. We assume for simplicity that the profile of valuations is such that $u_1 > u_2 > u_3 \geq u_i$ for all $i \geq 4$.
a) Find all Nash equilibria.
b) Find all dominated strategies of all players and all Nash equilibria in undominated strategies.
c) Is the game dominance-solvable?

Problem 6 *tragedy of the commons*

A pasture produces 100 units of grass, and a cow transforms x units of grass into x units of meat (worth $\$x$), where $0 \leq x \leq 10$, i.e., a cow eats at most 10 units of grass. It cost \$2 to bring a cow to and from the pasture (the profit from a cow that stays at home is \$2). Economic efficiency requires to bring exactly 10 cows to the pasture, for a total profit of \$80. A single farmer owning many cows would do just that.

Our n farmers, each with a large herd of cows, can send any number of cows to the commons. If farmer i sends s_i cows, s_N cows will share the pasture and each will eat $\min\{\frac{100}{s_N}, 10\}$ units of grass.

a) Write the payoff functions and show that in any Nash equilibrium the total number s_N of cows on the commons is bounded as follows

$$50\frac{n-1}{n} - 1 \leq s_N \leq 50\frac{n-1}{n} + 1$$

b) Deduce that the commons will be overgrazed by at least 150% and at most 400%, depending on n, and that almost the entire surplus will be dissipated in equilibrium. (*Hint: start by assuming that each farmer sends at most one cow*).

Problem 7 *a public good provision game.*
The common benefit function is $B(s) = \max_j s_j$: a single contributor is enough. Examples include R&D, ballroom dancing (who will be the first to dance) and dragon slaying (a lone knight must kill the dragon). Costs are quadratic, so the payoff functions are

$$u_i(s) = \max_j s_j - \frac{1}{2\lambda_i} s_i^2$$

where λ_i is a positive parameter differentiating individual costs.
a) Show that in any Nash equilibrium, only one agent contributes.

b) Show that there are p such equilibria, where p is the number of players i such that

$$\lambda_i \geq \frac{1}{2} \max_j \lambda_j$$

c) Compute strictly dominated strategies for each player. For what profiles (λ_i) is our game (strictly) dominance-solvable?

Problem 8 *the lobbyist game*

The two lobbyists choose an 'effort' level $s_i, i = 1, 2$, measured in money (the amount of bribes distributed) and the indivisible prize worth $\$a$ is awarded randomly to one of them with probabilities proportional to their respective efforts (if the prize is divisible, no lottery is necessary). Hence the payoff functions

$$u_i(s) = a\frac{s_i}{s_1 + s_2} - s_i \text{ if } s_1 + s_2 > 0; u_i(0, 0) = 0$$

a) Compute the best reply functions and show there is a unique Nash equilibrium.

b) Perform the successive elimination of strictly dominated strategies, and check the game is not dominance-solvable. However, if we eliminate an arbitrarily small interval $[0, \varepsilon]$ from the strategy sets, the reduced game is dominance solvable.

c) Show that the Nash equilibrium (of the full game) is strongly stable.

Problem 9

Two players share a *well* producing x liters of water at a cost $C(x) = \frac{1}{2}x^2$. Player i requests x_i liters of water, and the cost $C(x_1 + x_2)$ of pumping the total demand of water is divided in proportion to individual demands: player i pays $x_i \frac{C(x_1+x_2)}{x_1+x_2}$.

Player i's utility for x_i liters of water at cost c_i is

$$v_i(x_i) = 84 \log(1 + x_i) - c_i$$

a) Write the normal form of the game where each player chooses independently how much water to request.

b) Compute the quantities $\min_{x_j} \max_{x_i} u_i(x_i, x_j)$ and find the Nash equilibrium outcome. Show it is unique.

c) Is the Nash equilibrium outcome Pareto optimal? If not, compute the outcome maximizing total utility and compute the welfare loss at the equilibrium outcome.

d) Perform the successive elimination of strictly dominated strategies and comment on the result.

Problem 10

a) Prove Proposition 25.

b) Prove the statement discussed two paragraphs later. Given a game $(N, S_i, u_i, i \in N)$ with finite strategy sets, we write $S_N^\infty = \cap_{t=1}^\infty S_N^t$ for the result of the successive elimination of strictly dominated strategies. Consider any finite decreasing sequence $R_N^t \subseteq S_N$ such that $R_N^0 = S_N$ and $R_i^{t+1} \subseteq \mathcal{U}_i(R_N^t)$ for all i and t. Then show that $(R_N^T)^\infty = S_N^\infty$.

c) Prove Proposition 31.

Problem 11

There are 10 locations with values $0 < a_1 < a_2 < ... < a$. Player i ($i = 1.2$) has $n_i < 10$ soldiers and must allocate them among the locations (no more then one soldier per location). The payoff at location p is a_p to the player whose soldier is unchallenged, and $-a_p$ to his opponent; if they both have a soldier at location p, or no one does, the payoff is 0. The total payoff of the game is the sum of all locational payoffs.

Show that this game has a unique equilibrium in dominant strategies. What if some a_p are equal?

Problem 12 *price competition*

The two firms have constant marginal cost $c_i, i = 1, 2$ and no fixed cost. They sell two substitutable commodities and compete by choosing a price $s_i, i = 1, 2$. The resulting demands for the 2 goods are

$$D_i(s) = (\frac{s_j}{s_i + 1})^{\alpha_i}$$

where $\alpha_i > 1$. Show that there is an equilibrium in dominant strategies and discuss its stability.

Problem 13 *examples of best reply dynamics*

a) We have a symmetric two player game with $S_i = [0, 1]$ and the common best reply function

$$br(s) = \min\{s + \frac{1}{2}, 2 - 2s\}$$

Show that we have three Nash equilibria, all of them locally unstable, even for the sequential dynamics.

b) We have three players, $S_i = \mathbb{R}$ for all i, and the payoffs

$$u_1(s) = -s_1^2 + 2s_1s_2 - s_2^2$$

$$u_2(s) = -9s_2^2 + 6s_2s_3 - s_3^2$$

$$u_3(s) = -16s_1^2 - 9s_2^2 - s_3^2 + 24s_1s_2 - 6s_2s_3 + 8s_1s_3$$

Show there is a unique Nash equilibrium and compute it. Show the sequential best reply dynamics where players repeatedly take turns in the order $1, 2, 3$ does not converge to the equilibrium, whereas the dynamics where they repeatedly take turns in the order $2, 1, 3$ does converge from any initial point. What about the myopic adjustment where each player uses his best reply at each turn?

Problem 14 *stability analysis in two symmetric games*

a) This symmetrical n-person game has the strategy set $S_i = [0, +\infty[$ for all i and the payoff function

$$u_1(s) = s_2s_3 \cdots s_n(s_1e^{-(s_1+s_2+\cdots+s_n)} - 1)$$

(other payoffs deduced by the symmetry of the game).

Find all dominated strategies if any, and all Nash equilibria (symmetric or not) in pure strategies. Is this a potential game? Discuss the stability of the best reply and sequential best reply dynamics in this game.
b) Answer the same questions as in a) for the following symmetric game with the same strategy sets:

$$u_1(s) = s_2 s_3 \cdots s_n (2e^{-(s_1+s_2+\cdots+s_n)} + s_1)$$

Problem 15
Consider the following N players game. The set of pure strategies for each player is $C_i = \{1, ..., N\}$, thus the game consists in each player announcing (simultaneously and independently) an integer between 1 and N. To each pair of players i, j corresponds a number $v_{ij} (= v_{ji})$, interpreted as the utility both players could derive from being together (note that v_{ij} can be negative). Players are together if and only if they announce the same number. Thus, the payoff to each player i is the sum of v_{ij} over all players j who announced the same number as i. Prove that this game is a potential game and find all Nash equilibria.

Problem 16 *Congestion (variant of Example 3)*

We have n agents who travel from A to B at the same time. Agent i can use a private road at cost $c_i = i$ (that does not depend upon other agents' actions), or use the public road. If k agents travel on that road, they each pay a congestion cost k.

a) Describe the Nash equilibrium outcome (or outcomes) of this game.

b) Is this (are these) equilibrium outcome (s) Pareto optimal? Does it maximize total surplus? If not, compute the fraction of the efficient surplus wasted in equilibrium.

c) Show this is a potential game. Discuss the stability of the equilibrium outcome(s). Is the game dominance solvable (strictly or weakly)?

d) Now the private costs c_i are arbitrary numbers s.t. $1 \leq c_i \leq n$. Answer questions *a*) and *c*) above.

Problem 17 *Cost sharing*

We have n agents labeled $1, 2, \cdots, n$, who want to send a signal from A to B. Agent i can send her message via a private carrier at cost $c_i = \frac{1}{i}$ (independently of other agents' choices), or use the public link. If k agents use the public link, they each pay $\frac{1}{k}$.

a) Show that there is one Pareto inferior Nash equilibrium outcome and one Pareto optimal one. Show that the game is a potential game. Discuss the stability of these equilibrium outcomes.

b) Variant: the public link costs $\frac{1+\varepsilon}{k}$ to each user, where ε is a small positive number. Show that the game is now strictly dominance solvable. Compute the inefficiency loss, i.e., the ratio of the total cost in equilibrium to the efficient (minimal) cost of sending all messages.

c) Now the private costs c_i are arbitrary numbers s.t. $0 \leq c_i \leq 1$. Find the Nash equilibrium or equilibria, discuss their efficiency and whether the game is a potential game, or is dominance solvable.

Problem 18 *more congestion games*

We generalize the congestion games of Example 21. Now each player chooses among *subsets* of a fixed finite set S, so that $s_i \subset 2^S$. The same congestion function $f_x(m)$ applies to each element x in S. The payoff to player i is

$$u_i(s) = \sum_{x \in s_i} f_x(n_x(s)) \text{ where } n_x(s) = |\{j \in N | x \in s_j\}|$$

Interpretation: each commuter chooses a different route (origin and destination) on a common road network represented by a non oriented graph. Her own delay is the sum of the delays on all edges of the network.
Show that this game is still a potential game.

Problem 19 *A different congestion game*
There are m men and n women who must choose independently which one of two discos to visit. Let n_a, n_b be the number of women choosing to visit respectively disco A and disco B, and define similarly m_a, m_b. Each player only cares about the number of visitors of the opposite gender at the disco he or she visits.
a) Assume first the following payoff functions:

$$u_i = n_x \text{ if } i \text{ is a man choosing disco } X; v_j = m_x \text{ if } j \text{ is a woman choosing disco } X$$

Men (resp. women) want to be in the disco with more women (resp. men). Discuss the Nash equilibria of the game and their stability (strong and weak). It will help to show first that this game is a potential game.
b) Now the strategies of the $m + n$ players are the same but the payoffs are:

$$u_i = n_x \text{ if } i \text{ is a man choosing disco } X; v_j = -m_x \text{ if } j \text{ is a woman choosing disco } X$$

Here men want to be in the disco with more women, while women seek the disco with fewer men (remember this is a theoretical example).
Discuss the Nash equilibria of the game and their stability (strong and weak). Show that this game is **not** a potential game.

Problem 20 *ordinal potential games*
Let σ be the sign function $\sigma(0) = 0, \sigma(z) = 1$ if $z > 0, = -1$ if $z < 0$. Call a game $\mathcal{G} = (N, S_i, u_i, i \in N)$ an *ordinal potential game* if there exists a real valued function P defined on S_N such that for all i and $s_{-i} \in S_{-i}$ we have

$$\sigma\{u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})\} = \sigma\{P(s_i, s_{-i}) - P(s_i', s_{-i})\} \text{ for all } s_i, s_i' \in S_i$$

a)Show that the following Cournot oligopoly game is an ordinal potential game. Firm i chooses a quantity s_i, and D^{-1} is the inverse demand function. Costs are linear and identical:

$$u_i(s) = s_i D^{-1}(s_N) - cs_i \text{ for all } i \text{ and all } s$$

b) Show that Proposition 33 still holds for ordinal potential games.

Problem 21 *Cournot duopoly with increasing or U-shaped returns*
In all three questions the duopolists have identical cost functions C.

a) The inverse demand is $D^{-1}(q) = (150 - q)_+$ and the cost is

$$C(q) = 120q - \frac{2}{3}q^2 \text{ for } q \leq 90; = 5,400 \text{ for } q \geq 90$$

Show that we have three equilibria.
b) The inverse demand is $D^{-1}(q) = (130 - q)_+$ and the cost is

$$C(q) = \min\{50q, 30q + 600\}$$

Compute the equilibrium outcomes.
c)The inverse demand is $D^{-1}(q) = (150 - q)_+$ and the cost is

$$C(q) = 2,025 \text{ for } q > 0; = 0 \text{ for } q = 0$$

Compute the equilibrium outcomes.

Problem 22 *Cournot oligopoly with linear demand and costs*
The inverse demand for total quantity q is

$$D^{-1}(q) = \overline{p}(1 - \frac{q}{\overline{q}})_+$$

where $\overline{p}$ is the largest feasible price and $\overline{q}$ the supply at which the price falls to zero. Each firm i has constant marginal cost c_i and no fixed cost.
a) If all marginal costs c_i are identical, show there is a unique Nash equilibrium, where all n firms are active if $\overline{p} > c$, and all are inactive otherwise.
b) If the marginal costs c_i are arbitrary and $c_1 \leq c_2 \leq \cdots \leq c_n$, let m be zero if $\overline{p} \leq c_1$ and otherwise be the largest integer such that

$$c_i < \frac{1}{m+1}(\overline{p} + \sum_1^i c_k)$$

Show that in a Nash equilibrium outcome, exactly m firms are active and they are the lowest cost firms.

Problem 23 *Hoteling competition in location*
The consumers are uniformly spread on $[0, 1]$, and each wants to buy one unit. Each firm charges the fixed exogenous price p and chooses its location s_i in the interval. Production is costless. Once locations are fixed, each consumer shops in the nearest store. The tie-breaking rule: the demand is split equally between all stores choosing the same location
a) Show that with two competing stores, the unique Nash equilibrium is that both locate in the center. Show the game is not dominance-solvable. However, it is dominance solvable if each firm must locate in one of the $n + 1$ points $0, \frac{1}{n}, \frac{2}{n}, \cdots, 1$.
b) Show that with three competing stores, the game has no Nash equilibrium.
c) Show that with four competing stores, the game has a Nash equilibrium. Is it unique?
d) What is the situation with five stores?

Problem 24 *Hoteling competition in location: probabilistic choice*
a) Two stores choose a location on the interval $[0, 100]$. Customers are uniformly distributed on this interval, with at most a unit demand, and will shop from the nearest store if at all. If the distance between a customer and the store is t, he will buy with probability $p(t) = \frac{2}{\sqrt{t+4}}$. Thus if a store is located at 0 and is the closest store to all customers in the interval $[0, x]$, it will get from these customers the revenue

$$r(x) = \int_0^x p(t)dt = 4\sqrt{x+4} - 8$$

Stores maximize their revenues. Analyze the competition between the two stores and compute their equilibrium locations. Compare them to the collusive outcome, namely the choice of locations maximizing the total revenue of the two stores.
b) Generalize the model of question a). Now $p(t)$ is unspecified and so is its primitive $r(t)$. We assume that p is continuous, strictly positive, and strictly decreasing from $p(0) = 1$.
Under what condition on p do both stores locate at the midpoint in the Nash equilibrium of the game?
Show that if in equilibrium the stores choose different locations, they will never locate on $[0, 25]$ or $[75, 100]$.

Problem 25 *Hoteling competition in prices: two firms*
The 1000 consumers are uniformly spread on $[0, 3]$ and each wants to buy one unit and has a very large reservation price. The two firms produce costlessly and set arbitrary prices s_i. Once these prices are set consumers shop from the cheapest firm, taking into account the unit transportation cost t. A consumer at distance d_i from firm i buys

from firm 1 if $s_1 + td_1 < s_2 + td_2$, from firm 2 if $s_1 + td_1 > s_2 + td_2$

(the tie-breaking rule does not matter)
a) If the firms are located at 0 and 3, show that there is a unique Nash equilibrium pair of prices. Is it strongly/sequentially stable?
b) If the firms are located at 1 and 2, show that there is no Nash equilibrium (*hint: check first that a pair of two different prices can't be an equilibrium*).

Problem 26 *Hoteling competition in prices: three firms*
The consumers are uniformly spread over the interval $[0, 3]$ and each wants to buy one unit of the identical good produced by the three firms. The firms are located respectively at $0, 1$ and 3 and they produce costlessly. The transportation cost is 1 per unit. As usual consumers shop at the firm where the sum of the price and the transportation cost is smallest.
a) Write the strategic form of the game where the three firms choose the prices s_1, s_2, s_3 respectively.
b) Show that the game has a unique Nash equilibrium and compute it.
c) Is the equilibrium computed in b) strongly/sequentially stable?

Problem 27 *price war*
Two duopolists (a la Bertrand) have zero marginal cost and capacity c. The demand d is inelastic, with reservation price $\overline{p}$. Assume $c < d < 2c$. We also fix a small positive constant ε ($\varepsilon < \frac{\overline{p}}{10}$).
The game is defined as follows. Each firm chooses a price $s_i, i = 1, 2$ such that $0 \leq s_i \leq \overline{p}$. If $s_i \leq s_j - \varepsilon$, firm i sells its full capacity at price s_i and firm j sells $d - c$ at price s_j. If $|s_i - s_j| < \varepsilon$ the firms split the demand in half and sell at their own price (thus ε can be interpreted as a transportation cost between the two firms). To sum up

$$\begin{aligned} u_1(s) &= cs_1 \text{ if } s_1 \leq s_2 - \varepsilon \\ &= (d-c)s_1 \text{ if } s_1 \geq s_2 + \varepsilon \\ &= \frac{d}{2}s_1 \text{ if } s_2 - \varepsilon < s_1 < s_2 + \varepsilon \end{aligned}$$

with a symmetric expression for firm 2.
Set $p^* = \frac{d-c}{c}\overline{p}$ and check that the best reply correspondence of firm 1 is

$$\begin{aligned} br_1(s_2) &= \overline{p} \text{ if } s_2 < p^* + \varepsilon \\ &= \{\overline{p}, p^*\} \text{ if } s_2 = p^* + \varepsilon \\ &= s_2 - \varepsilon \text{ if } s_2 > p^* + \varepsilon \end{aligned}$$

Show that the game has no Nash equilibrium, and that the sequential best reply dynamics describes a cyclical price war.

Problem 28 *Bertrand duopoly*
The firms sell the same commodities and have the same cost function $C(q)$, that is continuous and increasing. They compete by setting prices $s_i, i = 1, 2$. The demand function D is continuous and decreasing. The low price firm captures the entire demand; if the 2 prices are equal, the demand is equally split between the 2 firms. Hence the profit function for firm 1

$$u_1(s) = s_1 D(s_1) - C(D(s_1)) \text{ if } s_1 < s_2; = 0 \text{ if } s_1 > s_2$$

$$= \frac{1}{2}s_1 D(s_1) - C(\frac{D(s_1)}{2}) \text{ if } s_1 = s_2$$

and the symmetrical formula for firm 2.
a) Show that if s^* is a Nash equilibrium, then $s_1^* = s_2^* = p$ and

$$AC(\frac{q}{2}) \leq p \leq 2AC(q) - AC(\frac{q}{2})$$

where $q = D(p)$ and $AC(q) = \frac{C(q)}{q}$ is the average cost function.
b) Assume increasing returns to scale, namely AC is (strictly) decreasing. Show there is no Nash equilibrium $s^* = (p, p)$ where the corresponding production q is positive. Find conditions on D and AC such that there is an equilibrium with $q = 0$.

c) In this and the next question assume decreasing returns to scale, i.e., AC is (strictly) increasing. Show that if $s^* = (p, p)$ is a Nash equilibrium, then $p_- \leq p \leq p_+$ where p_- and p_+ are solutions of

$$p_- = AC(\frac{D(p_-)}{2}) \text{ and } p_+ = 2AC(D(p_+)) - AC(\frac{D(p_+)}{2})$$

Check that the firms have zero profit at (p_-, p_-) but make a positive profit at (p_+, p_+) if $p_- < p_+$. *Hint: draw on the same figure the graphs of* $D^{-1}(q), AC(\frac{q}{2})$ *and* $2AC(q) - AC(\frac{q}{2})$.
d) To prove that the pair (p_+, p_+) found in question c) really is an equilibrium we must check that the revenue function $R(p) = pD(p) - C(D(p))$ is non decreasing on $[0, p_+]$. In particular p_+ should not be larger than the monopoly price.

Assume $C(q) = q^2$, $D(p) = (\alpha - \beta p)_+$ and compute the set of Nash equilibrium outcomes, discussing according to the parameters α, β.

Problem 29

In the game $\mathcal{G} = (N, S_i, u_i, i \in N)$ we write

$$\alpha_i = \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}); \beta_i = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

and assume the existence for each player of a prudent strategy $\overline{s}_i$, namely $\alpha_i = \min_{s_{-i}} u_i(\overline{s}_i, s_{-i})$.
a) Assume $\alpha = (\alpha_i)_{i \in N}$ is a Pareto optimal utility profile: there exists $\widetilde{s} \in S_N$ such that

$$\alpha = u(\widetilde{s}) \text{ and for all } s \in S_N : \{u(s) \geq u(\widetilde{s})\} \Rightarrow u(s) = u(\widetilde{s})$$

Show that $\alpha = \beta$ and that any profile of prudent strategies is a Nash equilibrium. Then we speak of an *inessential* game.
b) Assume that the strategy sets S_i are all finite, and $\beta = (\beta_i)_{i \in N}$ is a Pareto optimal utility profile. Show that if each function u_i is one-to-one on S_N then the outcome $\widetilde{s}$ such that $\beta = u(\widetilde{s})$ is a Nash equilibrium. Give an example of a game with finite strategy sets (where payoffs are not one-to-one) such that β is Pareto optimal and yet the game has no Nash equilibrium.