Economics 370
Microeconomic Theory
Problem Set 4
Answer Key

1) *For the production function* $f(x_1, x_2) = 2\sqrt{x_1} + x_2^2$*:*

a) *show whether the marginal product of factor 1 (respectively, factor 2) increases, stays constant or decreases as the amount of that factor alone is varied.*

$$MP_1 = \frac{\partial}{\partial x_1}\left(2\sqrt{x_1} + x_2^2\right) = x_1^{-1/2}; \text{ and}$$

$$MP_2 = \frac{\partial}{\partial x_2}\left(2\sqrt{x_1} + x_2^2\right) = 2x_2.$$

The marginal product of factor 1 is clearly decreasing, and the marginal product of factor 2 is clearly increasing.

b) *Show that this does not satisfy the definition of increasing returns to scale, nor constant returns to scale nor decreasing returns to scale. [Hint: find a combination of inputs such that doubling the amount of both inputs will more than double the amount of output, and find another combination of inputs such that doubling the amount of both inputs will less than double the output.]*

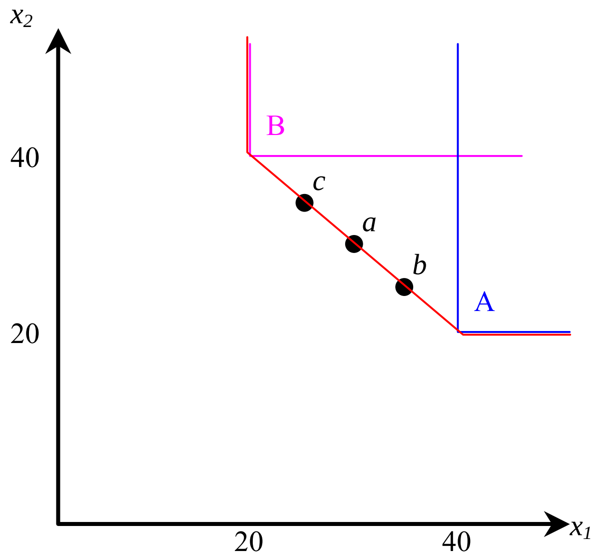
Consider the following combinations of inputs:

Input Combo	(1, 0)	(0, 1)
Output	2	1
Doubled:	(2, 0)	(0, 2)
Output	$2\sqrt{2} \approx 2.8$	8
Factor of Increase	≈ 1.4	4
Returns to Scale	Decreasing	Increasing

Since different combinations of inputs have increasing or decreasing returns to scale, the production function as a whole cannot be either increasing, decreasing or constant returns to scale.

2) *The Private Automobile Corporation has two plants for producing cars. Plant A produces according to* $f_A(x_1, x_2) = \min\{x_1, 2x_2\}$ *and plant B produces according to* $f_B(x_1, x_2) = \min\{2x_1, x_2\}$*, where* x_1 *and* x_2 *are the inputs.*

a) *Graph and label the isoquant for 40 cars at plant A. On the same diagram graph and label the isoquant for 40 cars at plant B.*



b) *Suppose that the firm wishes to produce 20 cars at each plant. How much of each input will the firm need to produce 20 cars at plant A, and how much of each input will the firm need to produce 20 cars at plant B. Label with an 'a' on your diagram, the point representing the total amount of each of the two inputs that the firm needs to produce a total of 40 cars, 20 at each plant.*

In order to produce 20 cars at plant A, the firm needs $x_1 = 20$ and $2x_2 = 20$.
That is, $x_1^A = 20$ and $x_2^A = 10$.

Similarly, $x_1^B = 10$ and $x_2^B = 20$.

Producing 40 cars by producing 20 at each plant requires:
$x_1^T = x_1^A + x_1^B = 20 + 10 = 30$; and
$x_2^T = x_2^A + x_2^B = 10 + 20 = 30$.

The point is shown on the graph above.

c) *Label with a b on your graph the point that shows how much of each of the inputs is needed altogether if the firm is to produce 10 cars at plant A and 30 cars at plant B. Label with a c on your graph the point that shows how much of each of the two inputs is needed in total if it is to produce 30 cars at plant A and 10 cars at plant B. Draw the firm's isoquant for producing 40 units of output (i.e. 40 cars) if it can split production in any manner between the two plants. Is the technology available to this firm convex?*

See the line drawn in red.

Clearly, the technology is convex.

3) *A firm has two factors and a production function* $f(x_1, x_2) = x_1^{1/2} x_2^{1/4}$. *The price of the output is 4. Factor 1 receives a wage of* w_1 *and factor 2 receives a wage of* w_1*. Solve for the (long-run) profit-maximizing choice of inputs* (x_1^*, x_2^*)*. How much output will it produce? How much profit will it make?*

One approach to solving this is based on the fact that at the optimal level of production:
$pMP_i = w_i$, for all factors.

$$MP_1 = \frac{\partial}{\partial x_1}\left(x_1^{1/2}x_2^{1/4}\right) = \frac{1}{2}x_1^{-1/2}x_2^{1/4}, \text{ and } MP_2 = \frac{\partial}{\partial x_2}\left(x_1^{1/2}x_2^{1/4}\right) = \frac{1}{4}x_1^{1/2}x_2^{-3/4}.$$

So, $4\frac{1}{2}x_1^{-1/2}x_2^{1/4} = w_1 \Rightarrow x_1^{1/2} = \frac{2}{w_1}x_2^{1/4} \Rightarrow x_1 = \frac{4}{w_1^2}x_2^{1/2}$.

And $4\frac{1}{4}x_1^{1/2}x_2^{-3/4} = w_2 \Rightarrow x_2^{3/4} = \frac{1}{w_2}x_1^{1/2} \Rightarrow x_2 = \frac{1}{w_2^{4/3}}x_1^{2/3}$

Then $x_1 = \frac{4}{w_1^2}x_2^{1/2} = \frac{4}{w_1^2}\left(\frac{1}{w_2^{4/3}}x_1^{2/3}\right)^{1/2} = \frac{4}{w_1^2 w_2^{2/3}}x_1^{1/3}$, so $x_1^{2/3} = \frac{4}{w_1^2 w_2^{2/3}}$, so $\boxed{x_1 = \frac{8}{w_1^3 w_2}.}$

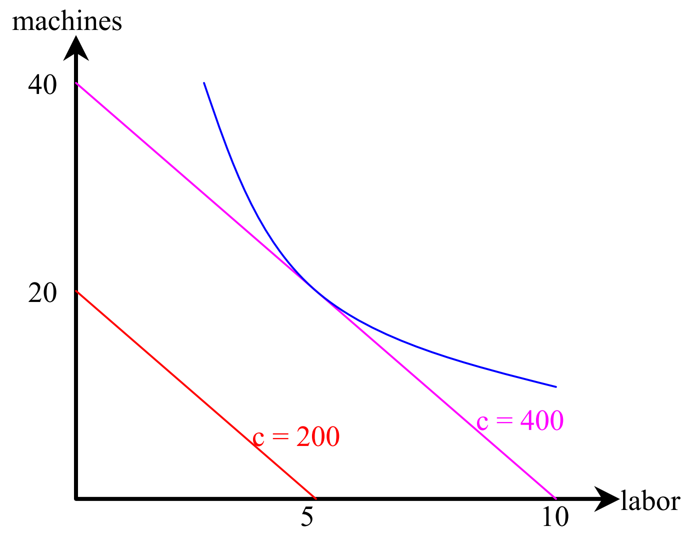
That means $x_2 = \frac{1}{w_2^{4/3}}x_1^{2/3} = \frac{1}{w_2^{4/3}}\left(\frac{8}{w_1^3 w_2}\right)^{2/3} = \frac{4}{w_1^2 w_2^{4/3} w_2^{2/3}} \boxed{= \frac{4}{w_1^2 w_2^2}.}$

Then $y = x_1^{1/2}x_2^{1/4} = \left(\frac{8}{w_1^3 w_2}\right)^{1/2}\left(\frac{4}{w_1^2 w_2^2}\right)^{1/4} = \frac{2^{3/2}2^{1/2}}{w_1^{3/2}w_2^{1/2}w_1^{1/2}w_2^{1/2}} = \frac{4}{w_1^2 w_2}$

Finally, $\pi = py - w_1x_1 - w_2x_2 = \frac{16}{w_1^2 w_2} - \frac{8w_1}{w_1^3 w_2} - \frac{4w_2}{w_1^2 w_2^2} = \frac{16}{w_1^2 w_2} - \frac{8}{w_1^2 w_2} - \frac{4}{w_1^2 w_2} = \frac{4}{w_1^2 w_2}$.

4) *A firm uses labor and machines to produce output according to the production function* $f(x_1, x_2) = 4\sqrt{x_1 x_2}$ *, where* x_1 *is the quantity of labor employed and* x_2 *is the quantity of machines used. The cost of labor is \$40 per unit and the cost of using a machine is \$10.*

a) *Draw an isocost line for this firm, showing combinations of labor and machines that cost \$400 and another isocost line showing combinations that cost \$200. What is the slope of these isocost lines?*



Slope = -4

b) *Suppose that the firm wants to produce its output in the cheapest possible way. Find the number of machines it would use per worker.*

Minimum cost production will satisfy:

$$TRS = \frac{p_1}{p_2}, \text{ or } \frac{MP_1}{MP_2} = \frac{p_1}{p_2} = \frac{40}{10} = 4.$$

Also,

$$MP_1 = \frac{\partial}{\partial x_1}\left(4\sqrt{x_1 x_2}\right) = 2\sqrt{\frac{x_2}{x_1}} \text{; and}$$

$$MP_2 = \frac{\partial}{\partial x_2}\left(4\sqrt{x_1 x_2}\right) = 2\sqrt{\frac{x_1}{x_2}}.$$

Therefore, $\frac{MP_1}{MP_2} = 2\sqrt{\frac{x_2}{x_1}} \cdot \frac{1}{2\sqrt{x_1/x_2}} = 2\sqrt{\frac{x_2}{x_1}} \cdot \frac{1}{2}\sqrt{\frac{x_2}{x_1}} = \frac{x_2}{x_1} = 4,$

so $x_2 = 4x_1$. That is, they would use 4 machines per worker.

c) *On the same graph you drew your two isocost lines, sketch the production isoquant corresponding to an output of 40. Calculate the amount of labor and the quantity of machines that are used to produce 40 units of output in the cheapest possible way, given the above factor prices. Calculate the cost of producing 40 units at these factor prices.*

The minimum-cost bundle for producing 40 units of output meet the requirement:

$$40 = 4\sqrt{x_1 x_2} = 4\sqrt{x_1(4x_1)} = 4\sqrt{4x_1^2} = 8x_1.$$

So, $x_1 = 5$, and $x_2 = 4 \cdot 5 = 20$.

Cost is $p_1 x_1 + p_2 x_2 = 40 \cdot 5 + 10 \cdot 20 = 400$.

A production isoquant is shown on the graph above. Note that it is tangent to the $400 isoquant at the minimum-cost production level.

d) *What amount of labor and what amount of machines would the firm use to produce y units of output in the cheapest possible way? How much would this cost?*

The minimum-cost bundle for producing y units of output meet the requirement:

$$y = 4\sqrt{x_1 x_2} = 4\sqrt{x_1(4x_1)} = 4\sqrt{4x_1^2} = 8x_1.$$

So, $x_1 = y / 8$; and $x_1 = y / 2$.

Cost is: $p_1 x_1 + p_2 x_2 = 40x_1 + 10x_2 = 40\frac{y}{8} + 10\frac{y}{2} = 5y + 5y = 10y.$

5) *Mary Magnolia wants to open a flower shop, the Petal Pusher, in a new mall. She has her choice of three different floor sizes, 200 square feet, 500 square feet, or 1,000 square feet. The monthly rent will be $1 a square foot. Mary estimates that if she has F square feet of floor space and sells y bouquets a month, her variable costs will be* $\$c_v(y) = y^2 / F$ *per month.*

a) *If she has 200 square feet of floor space, write down her marginal cost function and her average cost function. At what amount of output is average cost minimized?*

$$MC = \frac{\partial}{\partial y}\frac{y^2}{F} = \frac{2y}{F} = \frac{y}{100}.$$

$$ATC = \frac{c_v(y) + c_f}{y} = \frac{y^2}{yF} + \frac{F}{y} = \frac{y}{F} + \frac{F}{y} = \frac{y}{200} + \frac{200}{y}.$$

Average total cost is minimized where ATC = MC. So:

$\frac{y}{F} + \frac{F}{y} = \frac{2y}{F}$, or $y^2 + F^2 = 2y^2$, or $F^2 = y^2$, or $y = F = 200$.

b) *If she has 500 square feet, write down her marginal cost function and her average cost function. At what amount is average cost minimized?*

Using the equations derived above:

$$MC = \frac{y}{250}; \; ATC = \frac{y}{500} + \frac{500}{y}.$$

Average total cost is minimized at $y = F = 500$.

c) *If she has 1,000 square feet, write down her marginal cost function and her average cost function. At what amount is average cost minimized?*

Using the equations derived above:

$$MC = \frac{y}{500}; \; ATC = \frac{y}{1000} + \frac{1000}{y}.$$

Average total cost is minimized at $y = F = 1,000$.

d) *Illustrate in a diagram, Mary's (short-run) average cost curve and her (short-run) marginal cost curve if she has 200, 500 or 1,000 square feet of floor space. On the same diagram draw Mary's long-run average cost and long-run marginal cost curves.*