Math 102 Spring 2008: **Solutions: HW #6**

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section 10.4, #2 Given $f(x) = \sin x$ and $n = 4$, we have

$$\begin{aligned} f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x \end{aligned}$$

Therefore $P_4(x) = x - \frac{x^3}{3!}$ and $R_4(x) = \frac{x^5}{5!} \cos z$
for some number z between 0 and x.

section 10.4, #4 Given $f(x) = (1 - x)^{-1}$ and $n = 4$, we have

$$\begin{aligned} f'(x) &= (1-x)^{-2} & f'(0) &= 1 \\ f''(x) &= 2(1-x)^{-3} & f''(0) &= 2 \\ f^{(3)}(x) &= 6(1-x)^{-4} & f^{(3)}(0) &= 6 \\ f^{(4)}(x) &= 24(1-x)^{-5} & f^{(4)}(0) &= 24 \\ f^{(5)}(x) &= 120(1-x)^{-6} \end{aligned}$$

Therefore $P_4(x) = 1 + x + x^2 + x^3 + x^4$ and $R_4(x) = \frac{x^5}{(1-z)^6}$
for some number z between 0 and x.

section 10.4, #14 Given $f(x) = \sqrt{x}$, $a = 100$ and $n = 3$, we have

$$f(x) = \sqrt{x} \quad f(100) = 10$$

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} & f'(100) &= \frac{1}{20} \\ f''(x) &= \frac{-1}{4(x)^{\frac{3}{2}}} & f''(100) &= \frac{-1}{4000} \\ f^{(3)}(x) &= \frac{3}{8(x)^{\frac{5}{2}}} & f^{(3)}(0) &= \frac{3}{800000} \\ f^{(4)}(x) &= \frac{-15}{16(x)^{\frac{7}{2}}} \end{aligned}$$

Therefore $P_3(x) + R_3(x) = 10 + \frac{1}{20}(x - 100) + \frac{-1}{8000}(x - 100)^2 + \frac{3}{4800000}(x - 100)^3 + \frac{-15}{384(z)^{\frac{7}{2}}}(x - 100)^4$ for some number z between 100 and x.

section 10.4, #24 Since

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

then substitute x with x^3 in the series and get

$$\exp(x) = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \frac{x^{15}}{5!} + ... = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

section 10.4, #26 Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

then substiute x with $x/2$ and yeild

$$\sin\frac{x}{2} = \frac{x}{2} - \frac{x^3}{3! \cdot 8} + \frac{x^5}{5! \cdot 32} - \frac{x^7}{7! \cdot 128} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)! \cdot 2^{2n+1}}$$

section 10.4, #32 Given $f(x) = \sin x$ and $a = \pi/2$, we have

$$\begin{aligned} f(x) &= \sin x & f(a) &= 1 \\ f'(x) &= \cos x & f'(a) &= 0 \\ f''(x) &= -\sin x & f''(a) &= -1 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(a) &= 0 \\ f^{(4)}(x) &= \sin x & f^{(4)}(a) &= 1 \\ f^{(5)}(x) &= \cos x & f^{(5)}(a) &= 0 \\ f^{(5)}(x) &= -\sin x & f^{(6)}(a) &= -1 \end{aligned}$$

we conclude that $f^{(n)}(a) = 0$ if n is odd, whereas $f^{(n)}(a) = (-1)^{n/2}$ if n is even. So the Taylor series for $f(x)$ at a is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-\pi/2)^{2n} = 1 - \frac{1}{2}(x-\pi/2)^2 + \frac{1}{24}(x-\pi/2)^4 - \frac{1}{720}(x-\pi/2)^6 + ...$$

section 10.4, #36 Given $f(x) = 1/(1-x)^2$ and $a = 0$, we have

$$\begin{aligned} f(x) &= (1-x)^{-2} & f(a) &= 1 \\ f'(x) &= 2(1-x)^{-3} & f'(a) &= 2 \\ f''(x) &= 6(1-x)^{-4} & f''(a) &= 6 \\ f^{(3)}(x) &= 24(1-x)^{-5} & f^{(3)}(a) &= 24 \\ f^{(4)}(x) &= 120(1-x)^{-6} & f^{(4)}(a) &= 120 \\ f^{(5)}(x) &= 720(1-x)^{-7} & f^{(5)}(a) &= 720 \\ f^{(6)}(x) &= 5040(1-x)^{-8} & f^{(6)}(a) &= 5040 \end{aligned}$$

It is clear that $f^{(n)}(x) = (n+1)!$ for $n \geq 0$. Therefore the Taylor series for $f(x)$ at $a = 0$ is

$$\sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + ...$$

section 10.4, #42

$$\begin{aligned} D_x \cos x &= D_x(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + ...) \\ &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + ... \\ &= -\sin x \end{aligned}$$

and

$$\begin{aligned} D_x \sin x &= D_x(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + ...) \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - ... \\ &= \cos x \end{aligned}$$