

Math 102 Spring 2008: Solutions: HW #3

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1. section 7.5, #18 Evaluate

$$\int \frac{x+1}{x^3-x^2} dx.$$

We'll solve using partial fractions. If we assume $\frac{x+1}{x^3-x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$, clearing denominators gives us $Ax^2 - Ax + Bx - B + Cx^2 = x + 1$. Then associating like terms gives us that $A + C = 0$, $-A + B = 1$, and $-B = 1$. Therefore, $B = -1$, $A = -2$, and $C = 2$. Then we have

$$\frac{x+1}{x^3-x^2} dx = \int \left(\frac{2}{x-1} - \frac{2}{x} - \frac{1}{x^2} \right) dx = 2 \ln|x-1| - 2 \ln|x| + \frac{1}{x} + K.$$

2. section 7.5, #34 Evaluate

$$\int \frac{x^2+4}{(x^2+1)^2(x^2+2)} dx.$$

Again, we'll use partial fractions. If we assume that $\frac{x^2+4}{(x^2+1)^2(x^2+2)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$, then clearing denominators gives us that

$$\begin{aligned} A(x^5+2x^3+x) + B(x^4+2x^2+1) + C(x^5+3x^3+2x) + D(x^4+3x^2+2) \\ + E(x^3+2x) + F(x^2+2) \\ = (A+C)x^5 + (B+D)x^4 + (2A+3C+E)x^3 + (2B+3D+F)x^2 \\ + (A+2C+2E)x + (B+2D+2F) \\ = x^2 + 4 \end{aligned}$$

Therefore, we have that

$$\begin{aligned} A+C &= 0, & B+D &= 0, & 2A+3C+E &= 0 \\ 2B+3D+F &= 1, & A+2C+2E &= 0, & B+2D+2F &= 4 \end{aligned}$$

Solving for this system, we get that $A = 0$, $B = 2$, $C = 0$, $D = -2$, $E = 0$, and $F = 3$. Therefore,

$$\begin{aligned} \int \frac{x^2+4}{(x^2+1)^2(x^2+2)} dx &= \int \left(\frac{2}{x^2+2} - \frac{2}{x^2+1} + \frac{3}{(x^2+1)^2} \right) dx \\ &= \sqrt{2} \arctan\left(\frac{\sqrt{2}}{2}x\right) - \frac{1}{2} \arctan x + \frac{3}{2} \left(\arctan x + \frac{x}{x^2+1} \right) + C \end{aligned}$$

For the last integral on the right, we use trigonometric substitution, let $u = \tan \theta$, then

$$\begin{aligned} \int \frac{3}{(x^2 + 1)^2} dx &= \int \frac{3 \sec^2 \theta}{\sec^4 \theta} d\theta \\ &= 3 \int \frac{1}{\sec^2 \theta} d\theta \\ &= \frac{3}{2}(\theta + \sin \theta \cos \theta) \\ &= \frac{3}{2} \left(\arctan x + \frac{x}{x^2 + 1} \right) \end{aligned}$$

3. section 7.5, #38 Make a preliminary substitution before using the method of partial fractions to evaluate

$$\int \frac{\cos \theta}{\sin^2 \theta - \sin \theta - 6} dx$$

Let's use the substitution $u = \sin \theta$, so $du = \cos \theta d\theta$. Then we have

$$\frac{\cos \theta}{\sin^2 \theta - \sin \theta - 6} dx = \int \frac{1}{u^2 - u - 6} du$$

Then using partial fractions, let $\frac{1}{u^2 - u - 6} = \frac{A}{u - 3} + \frac{B}{u + 2}$, so we get that $(A + B)u + (2A - 3B) = 1$, and so $A = \frac{1}{5}$, $B = -\frac{1}{5}$. Then we have

$$\begin{aligned} \frac{\cos \theta}{\sin^2 \theta - \sin \theta - 6} dx &= \int \frac{1}{u^2 - u - 6} du \\ &= \frac{1}{5} \int \left(\frac{1}{u - 3} - \frac{1}{u + 2} \right) du \\ &= \frac{1}{5} (\ln |u - 3| - \ln |u + 2|) + C \\ &= \frac{1}{5} (\ln |\sin \theta - 3| - \ln |\sin \theta + 2|) + C \end{aligned}$$

4. section 7.6, #2 Use trigonometric substitution to evaluate

$$\int \frac{1}{\sqrt{4 - 9x^2}} dx$$

Let $x = \frac{2}{3} \sin u$. Then we have

$$4 - 9x^2 = 4 - 4 \sin^2 u = 4(1 - \sin^2 u) = 4 \cos^2 u$$

and $dx = \frac{2}{3} \cos u \, du$. This gives us that

$$\begin{aligned} \int \frac{1}{\sqrt{4-9x^2}} dx &= \frac{2}{3} \int \frac{1}{2 \cos u} \cos u \, du \\ &= \frac{1}{3} u + C \\ &= \frac{1}{3} \arcsin \frac{3x}{2} + C \end{aligned}$$

5. section 7.6, #12 Use trigonometric substitution to evaluate

$$\int \frac{x^3}{\sqrt{x^2+25}} dx$$

Let $x = 5 \tan u$, so we have that

$$x^2 + 25 = 25 \tan^2 u + 25 = 25 (\sec^2 u - 1) + 25 = 25 \sec^2 u$$

and $du = 5 \sec^2 u \, du$. This gives us that

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2+25}} dx &= \int \frac{125 \tan^3 u}{5 \sec u} \cdot 5 \sec^2 u \, du \\ &= 125 \int \tan^3 u \sec u \, du \\ &= 125 \int (\sec^2 u - 1) \tan u \sec u \, du \end{aligned}$$

Now make the substitution $w = \sec u$, so $dw = \sec u \tan u \, du$. Then we have

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2+25}} dx &= 125 \int (\sec^2 u - 1) \tan u \sec u \, du \\ &= 125 \int (w^2 - 1) dw \\ &= 125 \left(\frac{1}{3} w^3 - w \right) + C \\ &= 125 \left(\frac{1}{3} \sec^3 u - \sec u \right) + C \end{aligned}$$

Note that the right triangle with angle u , opposite side x , and adjacent side 5 has hypotenuse $\sqrt{x^2+25}$. Therefore, we have that $\sec u = \frac{\sqrt{x^2+25}}{5}$, so we get

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2+25}} dx &= 125 \left(\frac{1}{3} \cdot \frac{(x^2+25)^{3/2}}{125} - \frac{(x^2+25)^{1/2}}{5} \right) + C \\ &= (x^2+25)^{1/2} \left(\frac{1}{3} (x^2+25) - 25 \right) \\ &= \frac{1}{3} (x^2+25)^{1/2} (x^2-50) + C \end{aligned}$$

6. section 7.6, #16 Use trigonometric substitution to evaluate

$$\int \sqrt{1+4x^2} dx$$

Let $x = \frac{1}{2} \tan u$, so we have $dx = \frac{1}{2} \sec^2 u du$ and

$$\sqrt{1+4x^2} = \sqrt{1+\tan^2 u} = \sqrt{1+(\sec^2 u - 1)} = \sec u$$

Then we have

$$\begin{aligned} \int \sqrt{1+4x^2} dx &= \frac{1}{2} \int \sec^3 u du \\ &= \frac{1}{4} \sec u \tan u + \frac{1}{4} \ln |\sec u + \tan u| + C \end{aligned}$$

from integral formula for secant. Now note that the right triangle with angle u , opposite side $2x$, and adjacent side 1 has hypotenuse $\sqrt{1+4x^2}$, so $\sec u = \sqrt{1+4x^2}$ and $\tan u = 2x$, and so

$$\begin{aligned} \int \sqrt{1+4x^2} dx &= \frac{1}{4} \sec u \tan u + \frac{1}{4} \ln |\sec u + \tan u| + C \\ &= \frac{1}{2} x \sqrt{1+4x^2} + \frac{1}{4} \ln |\sqrt{1+4x^2} + 2x| + C \end{aligned}$$

7. section 7.6, #24 Use trigonometric substitution to evaluate

$$\int \frac{1}{(4-x^2)^2} dx$$

Let $x = 2 \sin u$, so $4-x^2 = 4-4\sin^2 u = 4\cos^2 u$ and $dx = 2 \cos u du$.

Then

$$\begin{aligned} \int \frac{1}{(4-x^2)^2} dx &= \int \frac{2 \cos u}{16 \cos^4 u} du \\ &= \frac{1}{8} \int \sec^3 u du \\ &= \frac{1}{16} \sec u \tan u + \frac{1}{16} \ln |\sec u + \tan u| + C \end{aligned}$$

Now consider the right triangle with angle u , opposite side x , and hypotenuse 2. This has adjacent side $\sqrt{4-x^2}$, so $\sec u = \frac{2}{\sqrt{4-x^2}}$ and $\tan u = \frac{x}{\sqrt{4-x^2}}$. Then we get that

$$\begin{aligned} \int \frac{1}{(4-x^2)^2} dx &= \frac{1}{16} \cdot \frac{2}{\sqrt{4-x^2}} \cdot \frac{x}{\sqrt{4-x^2}} + \frac{1}{16} \ln \left| \frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}} \right| + C \\ &= \frac{x}{8(4-x^2)} + \frac{1}{32} \ln \left| \frac{(2+x)^2}{(2+x)(2-x)} \right| + C \\ &= \frac{x}{8(4-x^2)} + \frac{1}{32} \ln \left| \frac{2+x}{2-x} \right| + C \end{aligned}$$

8. section 7.6, #36 Use trigonometric substitution to evaluate

$$\int (4x^2 - 5)^{3/2} dx$$

Let $x = \frac{\sqrt{5}}{2} \sec u$, so $dx = \frac{\sqrt{5}}{2} \sec u \tan u \, du$, and

$$4x^2 - 5 = 5 \sec^2 u - 5 = 5 \tan^2 u.$$

Therefore,

$$\begin{aligned} \int (4x^2 - 5)^{3/2} dx &= \int (5^{3/2} \tan^3 u) \cdot \frac{\sqrt{5}}{2} \sec u \tan u \, du \\ &= \frac{25}{2} \int \tan^4 u \sec u \, du \\ &= \frac{25}{2} \int (\sec^4 u - 2 \sec^2 u + 1) \sec u \, du \\ &= \frac{25}{2} \int (\sec^5 u - 2 \sec^3 u + \sec u) \, du \end{aligned}$$

From secant integral formula, we have

$$\begin{aligned} \int \sec^5 u \, du &= \frac{1}{4} \sec^3 u \tan u + \frac{3}{4} \int \sec^3 u \, du \\ &= \frac{1}{4} \sec^3 u \tan u + \frac{3}{8} \sec u \tan u + \frac{3}{8} \ln |\sec u + \tan u| + C \end{aligned}$$

Then

$$\begin{aligned} \int (4x^2 - 5)^{3/2} dx &= \frac{25}{2} \int (\sec^5 u - 2 \sec^3 u + \sec u) \, du \\ &= \frac{25}{2} \left(\frac{1}{4} \sec^3 u \tan u - \frac{5}{8} \sec u \tan u + \frac{3}{8} \ln |\sec u + \tan u| \right) + C \end{aligned}$$

Now consider the right triangle with angle u , adjacent side $\sqrt{5}$, and hypotenuse $2x$. It has opposite side $\sqrt{4x^2 - 5}$, and so $\sec u = \frac{2x}{\sqrt{5}}$ and $\tan u = \frac{\sqrt{4x^2 - 5}}{\sqrt{5}}$. This gives us

$$\begin{aligned} \int (4x^2 - 5)^{3/2} dx &= \frac{25}{2} \left(\frac{1}{4} \cdot \frac{8x^3}{5\sqrt{5}} \cdot \frac{\sqrt{4x^2 - 5}}{\sqrt{5}} - \frac{5}{8} \cdot \frac{2x\sqrt{4x^2 - 5}}{5} + \frac{3}{8} \ln \left| \frac{2x + \sqrt{4x^2 - 5}}{\sqrt{5}} \right| \right) + C \\ &= x^3 \sqrt{4x^2 - 5} - \frac{25}{8} x \sqrt{4x^2 - 5} + \frac{75}{16} \ln |2x + \sqrt{4x^2 - 5}| + C \end{aligned}$$

9. section 7.6, #44 Compute the arc length of the parabola $y = x^2$ over the interval $[0, 1]$.

With $y = x^2$, we have

$$ds = \sqrt{1 + (dy/dx)^2} dx = \sqrt{1 + 4x^2} dx$$

so we want to calculate $L = \int_0^1 \sqrt{1 + 4x^2} dx$. Therefore, let $x = \frac{1}{2} \tan \theta$, so then $1 + 4x^2 = 1 + \tan^2 \theta = \sec^2 \theta$, and $dx = \frac{1}{2} \sec^2 \theta$. Now we have

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + 4x^2} dx \\ &= \int_0^1 \sec \theta \cdot \frac{1}{2} \sec^2 \theta d\theta \\ &= \frac{1}{2} \int_0^1 \sec^3 \theta d\theta \\ &= \frac{1}{4} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \Big|_0^1 \\ &= \frac{1}{4} \left[2x\sqrt{1 + 4x^2} + \ln \left(2x + \sqrt{1 + 4x^2} \right) \right]_0^1 \\ &= \frac{1}{4} \left[2\sqrt{5} + \ln \left(2 + \sqrt{5} \right) \right] \end{aligned}$$

10. section 7.7, #6

The derivative of $\sqrt{3 - 2x - x^2}$ is $\frac{-1-x}{\sqrt{3-2x-x^2}}$. So

$$\int \frac{x+3}{\sqrt{3-2x-x^2}} dx = -\sqrt{3-2x-x^2} + \int \frac{2}{\sqrt{3-2x-x^2}} dx$$

Now we complete the square to get $3 - 2x - x^2 = 4 - (x+1)^2$. Substituting $u = x + 1$ we get

$$\int \frac{2}{\sqrt{3-2x-x^2}} dx = \int \frac{2du}{\sqrt{4-u^2}} = \int \frac{2du}{2\sqrt{1-(u/2)^2}}$$

Substituting $v = u/2$ we get

$$\int \frac{2dv}{\sqrt{1-v^2}} = 2 \sin^{-1}(v) + C$$

where to get this last equality we either used the substitution $v = \sin \theta$ or remembered the integral from memory.

Substituting back we get

$$2 \sin^{-1}(u/2) + C = 2 \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

so the final answer is

$$-\sqrt{3-2x-x^2} + 2 \sin^{-1}\left(\frac{x+1}{2}\right) + C.$$

11. section 7.7, #24

One can do this integral by integration by parts. But let's do it by substituting $x = \sin \theta$. We get

$$\int \frac{\sin^3 \theta}{(\cos^2 \theta)^4} \cos \theta d\theta = \int \frac{\sin^3 \theta}{\cos^7 \theta} d\theta.$$

Now $\sin^3 \theta = \sin \theta(1 - \cos^2 \theta)$ so we get

$$\int \frac{\sin \theta}{\cos^7 \theta} - \frac{\sin \theta}{\cos^5 \theta} d\theta$$

Now we substitute $u = \cos \theta$ to get

$$\int \frac{-du}{u^7} - \frac{-du}{u^5} = \frac{1}{6u^6} - \frac{1}{4u^4} + C$$

Substituting back we get

$$\frac{1}{6 \cos^6 \theta} - \frac{1}{4 \cos^4 \theta} + C = \frac{1}{6(\cos \sin^{-1} x)^6} - \frac{1}{4(\cos \sin^{-1} x)^4}$$

Now $\cos \sin^{-1}(x) = \sqrt{1 - x^2}$ so we get

$$\frac{1}{6(1 - x^2)^3} - \frac{1}{4(1 - x^2)^2}.$$

12. section 7.7, #36

$du = a \cos \theta d\theta$ so the left side equals

$$\begin{aligned} \int \frac{1}{(a^2 - a^2 \sin^2 \theta)^n} a \cos \theta d\theta &= \int \frac{1}{a^{2n} \cos^{2n} \theta} a \cos \theta d\theta \\ &= \frac{1}{a^{2n-1}} \int \frac{1}{\cos^{2n-1} \theta} d\theta \\ &= \frac{1}{a^{2n-1}} \int \sec^{2n-1} \theta d\theta \end{aligned}$$

13. section 7.8, #20

The function $\frac{x}{x^2-1}$ is undefined at $x = 1$ in the interval $[0, 2]$. Hence we must calculate $\int_0^1 \frac{x}{x^2-1} dx$ and $\int_1^2 \frac{x}{x^2-1} dx$. Now if $x > 1$ then $x^2 - 1 > 0$ so

$$\int \frac{x}{x^2-1} dx = \int \frac{du/2}{u} = 1/2 \ln(u) + C = 1/2 \ln(x^2 - 1) + C$$

where we substituted $u = x^2 - 1$. If $0 \leq x < 1$ then $x^2 - 1 < 0$ so

$$\int \frac{x}{x^2-1} dx = 1/2 \ln(1 - x^2) + C$$

Now as $b \rightarrow 1^-$ we have $1 - x^2 \rightarrow 0^+$ and hence $\ln(1 - x^2) \rightarrow -\infty$. Similarly, as $a \rightarrow 1^+$ we have $x^2 - 1 \rightarrow 0^+$ and hence $\ln(x^2 - 1) \rightarrow -\infty$. Hence

$$\begin{aligned} \int_0^1 \frac{x}{x^2-1} dx + \int_1^2 \frac{x}{x^2-1} dx &= \lim_{b \rightarrow 1^-} [1/2 \ln(1-x^2)]_0^b + \lim_{a \rightarrow 1^+} [1/2 \ln(x^2-1)]_a^2 \\ &= (-\infty) - 0 + (1/2 \ln(3) - (-\infty)) \end{aligned}$$

So the integral does not converge.

14. section 7.8 #32

$\cos x \rightarrow 0$ as $x \rightarrow \pi/2$ so the problem is at $x = \pi/2$. Now letting $u = \cos x$ we get

$$\int \frac{\sin x}{(\cos x)^{4/3}} dx = \int \frac{-du}{u^{4/3}} = 3u^{-1/3} + C = 3(\cos x)^{-1/3} + C.$$

Now as $x \rightarrow \pi/2^-$ we have $\cos(x) \rightarrow 0^+$ and $(\cos x)^{-1/3} \rightarrow \infty$. Thus the integral does not converge.

15. section 7.8 #52

$1 + x^2 > 0$ so we only have to figure out what happens as $x \rightarrow \pm\infty$. Now

$$\int \frac{1+x}{1+x^2} dx = \tan^{-1}(x) + \frac{1}{2} \ln(1+x^2) + C$$

which does not converge since $\ln(1+x^2) \rightarrow \infty$ as $x \rightarrow \infty$.

On the other hand

$$\begin{aligned} \int_{-t}^t \frac{1+x}{1+x^2} dx &= [\tan^{-1}(x) + \frac{1}{2} \ln(1+x^2)]_{-t}^t \\ &= \tan^{-1}(t) - \tan^{-1}(-t) + \frac{1}{2} \ln(1+t^2) - \frac{1}{2} \ln(1+(-t)^2) \\ &= \tan^{-1}(t) - \tan^{-1}(-t) \end{aligned}$$

Now as $t \rightarrow \infty$ we have $\tan^{-1}(t) \rightarrow \pi/2$ and $\tan^{-1}(-t) \rightarrow -\pi/2$ which means

$$\lim_{t \rightarrow \infty} \int_{-t}^t \frac{1+x}{1+x^2} dx = \pi/2 - (-\pi/2) = \pi.$$