Math 102 Spring 2008: **Solutions: HW #12**

Instructor: D. O'Donnol

1. section 9.3, #18

   Given the polar equation $r = 3\sin 3\theta$, we find that $r = 0$ when θ is any integral multiple of $\pi/3$. Hence the area of one loop is

$$A = \frac{1}{2}\int_0^{\frac{\pi}{3}} 9\sin^2 3\theta d\theta = \frac{9}{4}\int_0^{\frac{\pi}{3}} (1-\cos 6\theta)d\theta = \frac{3\pi}{4}.$$

2. section 9.3, #24

   The area of one loop this rose is

$$A = \frac{1}{2}\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} 36\cos^2 6\theta d\theta = 36\int_0^{\frac{\pi}{12}} \frac{1}{2}(1+\cos 12\theta)d\theta = \frac{3\pi}{2}.$$

3. section 9.3 #28

   Let A be the area of the region that is both inside the limaçon with polar equation $r = 2+\cos\theta$ and outside the circle with equation $r = 2$. The curves cross where $2+\cos\theta = 2$, thus where $\cos\theta = 0$; that is , where $\theta = \pm\pi/2$. Hence

$$\begin{aligned}A &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4+4\cos\theta+\cos^2\theta-4)d\theta\\ &= \int_0^{\frac{\pi}{2}} (4\cos\theta + \frac{1+\cos 2\theta}{2})d\theta\\ &= 4+\frac{\pi}{4}.\end{aligned}$$

4. section 9.3, #32

   See Fig.9.3.17 of the text. Given $r = 1-2\sin\theta$, we see that $r = 0$ when $\sin\theta = \frac{1}{2}$; that is, when $\theta = \frac{\pi}{6}$ and when $\theta = \frac{5\pi}{6}$. The small loop is formed when $\frac{1}{6}\pi \le \theta \le \frac{5}{6}\pi$, where $r \le 0$. Let A_2 denote its area. The large loop is formed when $\frac{5}{6}\pi \le \theta \le \frac{13}{6}\pi$, where $r \le 0$. Let A_1 denote its area. Also note that

$$\begin{aligned}\frac{1}{2}(1-2\sin\theta)^2 &= \frac{1}{2}(1-4\sin\theta+4\sin^2\theta)\\ &= \frac{1}{2}-2\sin\theta+1-\cos 2\theta\\ &= \frac{3}{2}-2\sin\theta-\cos 2\theta.\end{aligned}$$

   Therefore,

$$\begin{aligned}A_1 &= \int_{\frac{5\pi}{6}}^{\frac{13\pi}{6}} (\frac{3}{2}-2\sin\theta-\cos 2\theta)d\theta\\ &= [\frac{3}{2}\theta+2\cos\theta-\frac{1}{2}\sin 2\theta]_{5\pi/6}^{13\pi/6}\\ &= \frac{3\sqrt{3}+4\pi}{2}.\end{aligned}$$

   and

$$\begin{aligned}A_2 &= [\frac{3}{2}\theta+2\cos\theta-\frac{1}{2}\sin 2\theta]_{\pi/6}^{5\pi/6}\\ &= \frac{-3\sqrt{3}+2\pi}{2}.\end{aligned}$$

Because A_1 measures all of the area within the large loop–including that within the small loop–the area that is both within the large loop of the limaçon and outside its small loop is

$$A = A_1 - A_2 = \pi + 3\sqrt{3}.$$

5. section 9.3, #38

   The circles $r = 1$ and $r = 2\cos\theta$ meet where $\theta = \pi/3$; the circles $r = 1$ and $r = 2\sin\theta$ meet where $\theta = \pi/6$. Hence the area of the region that lies within all three circles is

$$\begin{aligned} A &= \frac{1}{2}[\int_0^{\frac{\pi}{6}} (2\sin\theta)^2 d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1^2 d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (2\cos\theta)^2 d\theta] \\ &= \frac{1}{2}[\int_0^{\frac{\pi}{6}} 2(1-\cos 2\theta)d\theta + \frac{\pi}{6} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2(1+\cos 2\theta)d\theta] \\ &= \frac{5\pi - 6\sqrt{3}}{12}. \end{aligned}$$

6. section 9.4, #8

   If $x = 2e^t$, then $y = 2e^{-t} = \frac{4}{2e^t} = \frac{4}{x}, x > 0$.

7. section 9.4, #18

   (a) At $t = \pi/4$, $(x, y) = (\sqrt{2}/4, \sqrt{2}/4)$. The slope of the tangent line is

   $$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3(\sin^2 x)(\cos x)}{3(\cos^2 x)(-\sin x)} = -\frac{\sin x}{\cos x} = -\tan x.$$

   So at $t = \pi/4$ the slope is $-\tan \pi/4 = -1$. By the point slope formula the tangent line is

   $$y - \sqrt{2}/4 = -(x - \sqrt{2}/4).$$

   (b) Now,

   $$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\sec^2 x}{3(\cos^2 x)(-\sin x)} = \frac{1}{3}\sec^4 x \csc x.$$

   So at $t = \pi/4$,

   $$\frac{d^2y}{dx^2} = \frac{1}{3}\sqrt{2}^5 = \frac{\sqrt{2}}{3} > 0$$

   so it is concave up at $t = \pi/4$.

8. section 9.4, #26

   (a) The tangent line is horizontal if $\frac{dy}{dx} = 0$. Here $\frac{dy}{dx} = 2\cos 2t\cos t$, so this is zero when $2\cos 2t = 0$, so $t = \frac{2n-1\pi}{4}$ where n is an integer. So the graph have a horizontal tangent line at $(\pm\frac{1}{2}\sqrt{2}, \pm 1)$.

   (b) The graph crosses the x-axis when $0 = x = \sin 2t$, that is when $t = 0, \pi/2, \pi, 3\pi/2$. When $t = 0$ it passes through the origin with slope 2, and when $t = \pi$ it passes through the origin with slope -2, so the tangent line at the origin is not well defined. The other two intercept points are $(1, 0)$ and $(0, 1)$ where the tangent lines are vertical.

9. section 9.5, #2

   The area is

   $$\begin{aligned} \int_0^{\ln 2} (e^{-t}) \cdot (3e^{3t})dt &= \int_0^{\ln 2} 3e^{2t}dt \\ &= [\frac{3}{2}e^{2t}]_0^{\ln 2} \\ &= \frac{3}{2}(e^{2\ln 2} - e^0) = \frac{3}{2}(2^2 - 1) = \frac{9}{2} \end{aligned}$$

10. section 9.5, #12

    The arclength is

    $$\begin{aligned} \int_0^1 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 dt} &= \int_0^1 \sqrt{(t)^2 + (t^2)^2}dt \\ &= \int_0^1 t\sqrt{1 + t^2}dt \\ &= [\frac{1}{3}(1 + t^2)^{3/2}]_0^1 = \frac{1}{3}(2^{3/2} - 1) \end{aligned}$$

11. section 9.5, #16

    For $r = \theta$ we have $ds = \sqrt{1 + \theta^2}$. Next use substitution and let $\theta = \tan u$ then $d\theta = \sec^2 u du$.

$$
\begin{aligned}
\int_{2\pi}^{4\pi} \sqrt{1+\theta^2}d\theta &= \int_a^b \sqrt{1+\tan^2 u}(\sec^2 u)du \\
&= \int_a^b \sec^3 u du \\
&= [\frac{1}{2}(\sec u)(\tan u) + \frac{1}{2}\ln|\sec u + \tan u|]_a^b
\end{aligned}
$$

Then use the triangle given by $\theta = \tan u$ to see $\sec u = \sqrt{1+\theta^2}$ so

$$
\begin{aligned}
[\frac{1}{2}(\sec u)(\tan u) + \frac{1}{2}\ln|\sec u + \tan u|]_a^b &= [\frac{\theta}{2}\sqrt{\theta^2+1} + \frac{1}{2}\ln|\theta + \sqrt{\theta^2+1}|]_{2\pi}^{4\pi} \\
&= 2\pi\sqrt{16\pi^2+1} + \frac{1}{2}\ln|4\pi + \sqrt{16\pi^2+1}| \\
&\quad -\pi\sqrt{4\pi^2+1} - \frac{1}{2}\ln|2\pi + \sqrt{4\pi^2+1}|.
\end{aligned}
$$