Math 102 Spring 2008: **Solutions: HW #12**

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1. section 9.3, #18

   Given the polar equation $r = 3\sin 3\theta$, we find that $r = 0$ when θ is any integral multiple of $\pi/3$. Hence the area of one loop is

$$A = \frac{1}{2}\int_0^{\frac{\pi}{3}} 9\sin^2 3\theta d\theta = \frac{9}{4}\int_0^{\frac{\pi}{3}} (1-\cos 6\theta)d\theta = \frac{3\pi}{4}.$$

2. section 9.3, #24

   The area of one loop this rose is

$$A = \frac{1}{2}\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} 36\cos^2 6\theta d\theta = 36\int_0^{\frac{\pi}{12}} \frac{1}{2}(1+\cos 12\theta)d\theta = \frac{3\pi}{2}.$$

3. section 9.3 #28

   Let A be the area of the region that is both inside the limaçon with polar equation $r = 2+\cos\theta$ and outside the circle with equation $r = 2$. The curves cross where $2+\cos\theta = 2$, thus where $\cos\theta = 0$; that is , where $\theta = \pm\pi/2$. Hence

$$\begin{aligned} A &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4+4\cos\theta+\cos^2\theta-4)d\theta \\ &= \int_0^{\frac{\pi}{2}} (4\cos\theta + \frac{1+\cos 2\theta}{2})d\theta \\ &= 4+\frac{\pi}{4}. \end{aligned}$$

4. section 9.3, #32

   See Fig.9.3.17 of the text. Given $r = 1-2\sin\theta$, we see that $r = 0$ when $\sin\theta = \frac{1}{2}$; that is, when $\theta = \frac{\pi}{6}$ and when $\theta = \frac{5\pi}{6}$. The small loop is formed when $\frac{1}{6}\pi \le \theta \le \frac{5}{6}\pi$, where $r \le 0$. Let A_2 denote its area. The large loop is formed when $\frac{5}{6}\pi \le \theta \le \frac{13}{6}\pi$, where $r \le 0$. Let A_1 denote its area. Also note that

$$\begin{aligned} \frac{1}{2}(1-2\sin\theta)^2 &= \frac{1}{2}(1-4\sin\theta+4\sin^2\theta) \\ &= \frac{1}{2} - 2\sin\theta + 1 - \cos 2\theta \\ &= \frac{3}{2} - 2\sin\theta - \cos 2\theta. \end{aligned}$$

   Therefore,

$$\begin{aligned} A_1 &= \int_{\frac{5\pi}{6}}^{\frac{13\pi}{6}} (\frac{3}{2} - 2\sin\theta - \cos 2\theta)d\theta \\ &= [\frac{3}{2}\theta + 2\cos\theta - \frac{1}{2}\sin 2\theta]_{5\pi/6}^{13\pi/6} \\ &= \frac{3\sqrt{3}+4\pi}{2}. \end{aligned}$$

   and

$$\begin{aligned} A_2 &= [\frac{3}{2}\theta + 2\cos\theta - \frac{1}{2}\sin 2\theta]_{\pi/6}^{5\pi/6} \\ &= \frac{-3\sqrt{3}+2\pi}{2}. \end{aligned}$$

Because A_1 measures all of the area within the large loop–including that within the small loop–the area that is both within the large loop of the limaçon and outside its small loop is

$$A = A_1 - A_2 = \pi + 3\sqrt{3}.$$

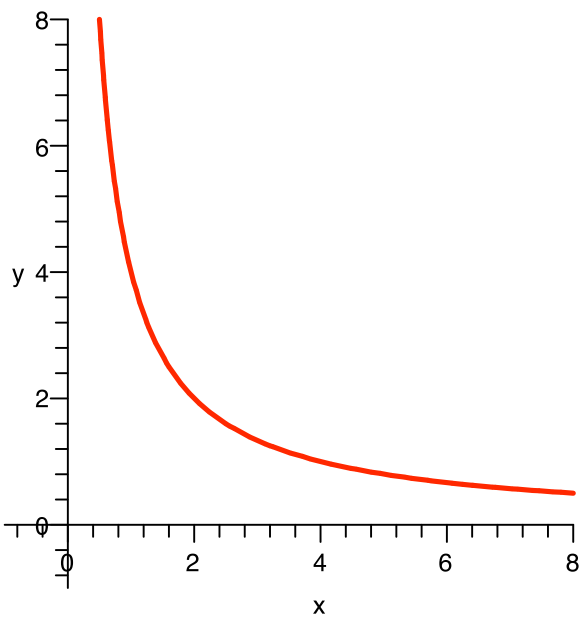
5. section 9.3, #38

   The circles $r = 1$ and $r = 2\cos\theta$ meet where $\theta = \pi/3$; the circles $r = 1$ and $r = 2\sin\theta$ meet where $\theta = \pi/6$. Hence the area of the region that lies within all three circles is

$$\begin{aligned} A &= \frac{1}{2}[\int_0^{\frac{\pi}{6}} (2\sin\theta)^2 d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1^2 d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (2\cos\theta)^2 d\theta] \\ &= \frac{1}{2}[\int_0^{\frac{\pi}{6}} 2(1 - \cos 2\theta) d\theta + \frac{\pi}{6} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2(1 + \cos 2\theta) d\theta] \\ &= \frac{5\pi - 6\sqrt{3}}{12}. \end{aligned}$$

6. section 9.4, #8

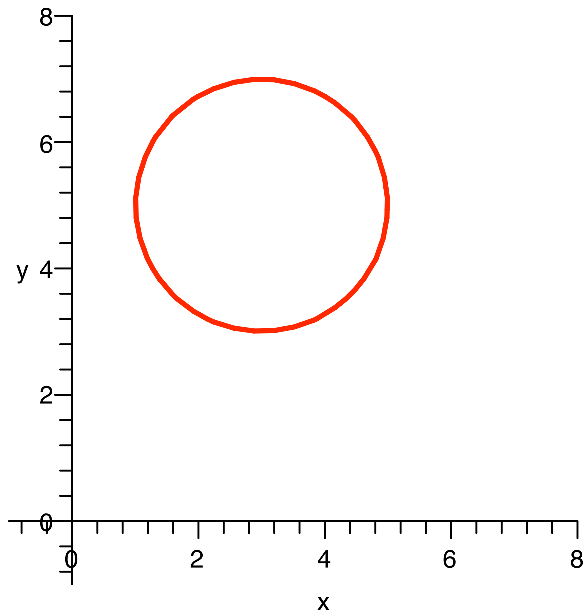
   If $x = 2e^t$, then $y = 2e^{-t} = \frac{4}{2e^t} = \frac{4}{x}, x > 0$.



7. section 9.4, #14 Given $x = 3 + 2\cos t$ and $y = 5 - 2\sin t, 0 \le t \le 2\pi$, we find that

$$(\frac{x-3}{2})^2 + (\frac{y-5}{2})^2 = \cos^2 t + \sin^2 t = 1$$

so that $(x-3)^2 + (y-5)^2 = 4$. The graph is a circle of radius 2 with center at (3,5). As t varies from 0 to 2π, the point (x,y) begins at the point (5,5) and moves once clockwise around the circle.



8. section 9.5, #2

The area is

$$\begin{aligned} \int_0^{\ln 2} (e^{-t}) \cdot (3e^{3t}) dt &= \int_0^{\ln 2} 3e^{2t} dt \\ &= [\frac{3}{2} e^{2t}]_0^{\ln 2} \\ &= \frac{3}{2}(e^{2\ln 2} - e^0) = \frac{3}{2}(2^2 - 1) = \frac{9}{2} \end{aligned}$$

9. section 9.5, #12

The arclength is

$$\begin{aligned} \int_0^1 \sqrt{(x')^2 + (y')^2} dt &= \int_0^1 \sqrt{(t)^2 + (t^2)^2} dt \\ &= \int_0^1 t\sqrt{1 + t^2} dt \\ &= [\frac{1}{3}(1 + t^2)^{3/2}]_0^1 = \frac{1}{3}(2^{3/2} - 1) \end{aligned}$$

10. section 9.5, #16

In parametrized form $r = \theta$ takes the form $x = \theta\cos\theta$ and $y = \theta\sin\theta$. Thus the arclength is

$$\begin{aligned}
\int_{2\pi}^{4\pi} \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2}d\theta &= \int_{2\pi}^{4\pi} \sqrt{1+\theta^2}d\theta \\
&= [\frac{\theta}{2}\sqrt{\theta^2+1} + \frac{1}{2}\ln|\theta + \sqrt{\theta^2+1}|]_{2\pi}^{4\pi} \\
&= 2\pi\sqrt{16\pi^2+1} + \frac{1}{2}\ln|4\pi + \sqrt{16\pi^2+1}| \\
&\quad -\pi\sqrt{4\pi^2+1} - \frac{1}{2}\ln|2\pi + \sqrt{4\pi^2+1}|
\end{aligned}$$

where I looked up the integral $\int \sqrt{1+\theta^2}d\theta$ in a table to obtain the second equality.