

Rice University
Physics 332

MEASUREMENT OF THE GRAVITATIONAL
CONSTANT

I. INTRODUCTION2
II. ANALYSIS.....3
III. EXPERIMENT.....8

May 1998

I. Introduction

Newton's law of gravitation, $F = GmM/r^2$, describes the attraction between any two masses in the universe. The constant G is obviously fundamental, in the same sense as the electron charge or Planck's constant. It is perhaps surprising, therefore, that after two centuries of effort G is only known with an accuracy of about 100 ppm, whereas most of the other constants have been measured to 1 ppm or better. The reason for this discrepancy is, of course, that gravity is the weakest of the fundamental forces and a measurement of G is correspondingly difficult. For example, the gravitational force between two 1 kg lead spheres which nearly touch is about 2×10^{-9} of the weight of either one. In the present exercise you can measure G and estimate the errors, thereby learning how one might go about measuring such a small effect with reasonable accuracy.

A determination of G is conceptually easy: Measure the force between two known masses arranged in a known geometry and compare the result with an appropriate estimate of mM/r^2 . Determining the geometry and doing the calculation both require care, but the real challenge is in measuring the force. The device used to do this is a torsion balance, invented by one Rev. John Michell about 1750 and independently by Augustin Coulomb in 1785. It has been refined and applied to the measurement of G by numerous workers, beginning with Henry Cavendish in 1798. The best current estimate of G is $(6.67259 \pm 0.00085) \times 10^{-8}$ dynes $\text{cm}^2 \text{g}^{-2}$. A fairly extensive bibliography of the various experiments that have led to this value can be found in an article by G. T. Gillies, *American Journal of Physics* **58**, 525 (1990).

The analysis of this experiment becomes rather messy because of the need to keep track of many numeric parameters. You should have some skill in using at least a programmable calculator, and preferably a computer, if you wish to undertake this experiment.

II. Analysis

A simplified torsion balance is shown in Fig. 1. When the large masses M are in place, the gravitational force on the small masses causes the beam to rotate, twisting the suspension fiber. The deformed fiber exerts a restoring torque so the assembly comes to rest after rotating through an angle determined by the magnitude of the gravitational attraction. The angle of rotation can be measured with high sensitivity by noting the deflection of the light beam on a distant scale. Repeating the deflection measurement with the large masses on the opposite sides further increases the sensitivity. Knowing the deflection angles and the parameters of the system allows us to determine G .

The basic relation between the deflection θ and the force is

$$\theta = \tau/\kappa \quad (1)$$

where κ is the torsion constant of the fiber and τ is the vertical component of the net gravitational torque. The torsion constant is found by determining the frequency of torsional oscillations of the pendulum when the large masses are absent. The ratio $E = \tau/G$ can be calculated from the known masses and geometry and Equation 1 can then be solved to give G in terms of known quantities

$$G = \frac{\kappa\theta}{E} \quad (2)$$

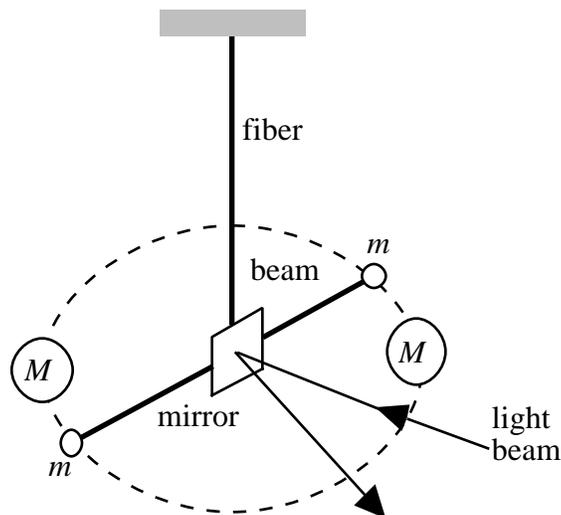


Fig. 1 Schematic of torsion pendulum showing the two masses, suspension fiber and optical lever arrangement.

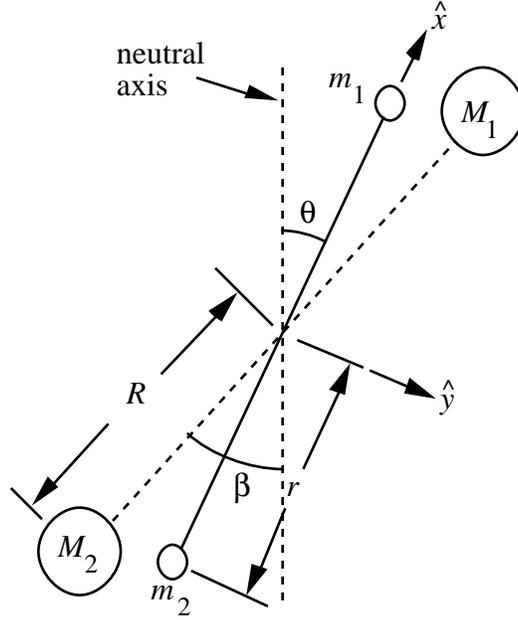


Fig. 2 Plan view of torsion pendulum, showing angles and lengths. The axes indicated are convenient for calculating torques about the center.

We turn now to the details of finding these quantities, using the parameters defined in Fig. 2 and Table 1.

The largest gravitational torques are produced by the large masses M acting on each of the small masses m . The placement of the large masses is symmetrical, so we only need to calculate

$$\vec{\tau}_{11} = \vec{r}_{m1} \times \vec{F}_{11} \quad \text{and} \quad \vec{\tau}_{12} = \vec{r}_{m2} \times \vec{F}_{21} \quad (3)$$

where \vec{r}_{mi} is the position vector of m_i . Since the masses are spherical, the force of M_i on m_j is given by

$$\vec{F}_{ij} = \frac{Gm_iM_j}{r_{ij}^3} \vec{r}_{ij} \quad (4)$$

where \vec{r}_{ij} is the vector from m_i to M_j . Equations 3 and 4 can be evaluated in terms of the parameters listed in Table 1 and then divided by G to give this contribution to E . There will be an equal contribution from the other mass M .

Table 1 Apparatus parameters and uncertainties.

Parameter	Value	Uncertainty	Description
β	0.750	0.001	Angular position of M
r	5.04 cm	0.03 cm	Pivot to center spacing of m
R	6.890 cm	0.011 cm	Pivot to center spacing of M
m	15.1 gm	0.1 gm	Mass of m
M	1500 gm	1.0 gm	Mass of M
r_m	0.69 cm	0.01 cm	Radius of m
ℓ_b	4.28 cm	0.03 cm	Half-length of beam
m_b	1.10 gm	0.07 gm	Mass of beam

The attractive force between M and the supporting beam is smaller, but not entirely negligible. Assume the beam has length $2\ell_b$ and mass m_b . A small part of the beam of length dx located at \vec{r}_b has mass $dm = \mu dx$ and is attracted to M_1 with a force

$$d\vec{F}_{b1} = \frac{GM}{r_{b1}^3} \vec{r}_{b1} dm \quad (5)$$

The corresponding torque $d\vec{\tau}_{b1} = \vec{r}_{b1} \times d\vec{F}_{b1}$, and the total torque is found by integrating $d\tau$ over the length of the beam

$$\vec{\tau}_{b1} = \mu GM \int_{-\ell_b}^{\ell_b} \frac{\vec{r}_{b1} \times \vec{r}_b}{r_{b1}^3} dx \quad (6)$$

The integration can be done analytically and then numerically evaluated, or by direct numerical quadrature. As before, the value of E must be doubled to account for the other M .

So far we have assumed that the movable beam will be in equilibrium along the center line of the apparatus when the large masses are absent. It is far more likely that there will be a small offset, as indicated in Fig. 3. Recalling that the deflections θ are measured relative to the true neutral position, we deduce that θ will be different for clockwise and counter-clockwise rotations. The amount of the offset, $\delta\beta$ in Fig. 3, is not easy to measure directly but it can be inferred from the deflection data. Note that the effect of the offset is to increase β slightly for one orientation of the large masses, and decrease it by the same amount for the other. We can take account of this in the analysis by increasing β for one direction of deflection and decreasing it by

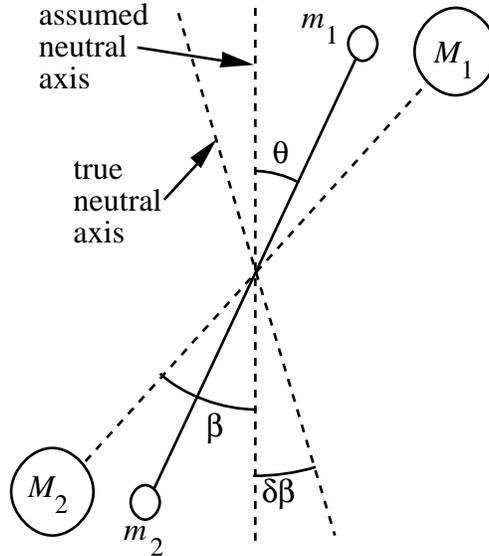


Fig. 3 Offset $\delta\beta$ between the assumed and true equilibrium positions of the beam. Note that deflections are necessarily measured from the true neutral axis.

the same amount for the other direction. The correct increment will lead to the same value of G for both clockwise and counter-clockwise deflections.

The torsion constant is found from the frequency and damping of free oscillations (M 's removed) of the torsion pendulum. The equation of motion for θ is

$$I \frac{d^2\theta}{dt^2} = -\kappa\theta - \Gamma \frac{d\theta}{dt} \quad (7)$$

where I is the moment of inertia of the beam and Γ is a damping factor. The solution to this is

$$\theta = Ae^{-\gamma t} \cos(\omega_1 t + \phi) \quad (8)$$

where

$$\omega_0 = \sqrt{\kappa/I}, \quad \gamma = \Gamma/2I \quad \text{and} \quad \omega_1 = (\omega_0^2 - \gamma^2)^{1/2} \quad (9)$$

Fitting Eq. 8 to a plot of $\theta(t)$ yields estimates of ω_1 and γ . These determine κ through

$$\kappa = I\omega_1^2 \left(1 + \frac{\gamma^2}{\omega_1^2} \right) \quad (10)$$

once I has been calculated from the known parameters of the beam.

Error analysis proceeds from the standard formula for uncorrelated uncertainties

$$\sigma_G^2 = \sum \left(\frac{\partial G}{\partial q} \right)^2 \sigma_q^2 \quad (11)$$

with terms for θ , ω_1 and γ as well as the parameters listed in Table 1. Equation 11 could be evaluated analytically, but that would be tedious. A simple numerical algorithm suffices for the derivatives: Find the change in G for a small, perhaps 1%, change in each q and approximate $(\partial G/\partial q)$ by $\Delta G/\Delta q$. Doing this for each q in turn will give the terms of the sum. If one were doing further development, this procedure would also show which parameters should be better known to improve accuracy.

III. Experiment

The torsion pendulum is mounted on a stand that can be leveled and that has holders to reproducibly position the large masses. The pertinent dimensions of this system are listed in Table 1. Removable shields protect the small masses from electrostatic fields and from radiative heating that might produce unwanted deflections. The angular deflection of the beam is measured with an optical lever arrangement, reflecting a laser beam from a mirror on the beam to a calibrated scale across the room. This section discusses the adjustment of the apparatus and subsequent data acquisition.

The suspension fiber is necessarily fragile. If you need to move or reorient the balance, the beam *must* be clamped with the two thumbscrews on the bottom of the balance case. Remove the shields so you can observe the beam and screw in the clamps until the beam is gently pinned to the top of the case. Failure to follow this precaution may destroy the suspension system. The balance will require about 12 hours to come to rest after release of the clamps.

The pendulum assembly should be positioned near one end of the lab bench, with the pendulum suspension rod centered in the holes in the clear convection baffles. If leveling is needed, small adjustments can be made with the three screws provided. It will take a few hours for the swing modes to damp out sufficiently for data to be taken, so it is desirable to avoid adjustment unless absolutely necessary.

The optical lever consists of a laser-illuminated slit, a concave mirror fixed to the beam and a scale at the far end of the lab bench. Position the laser so that the central part of the slit diffraction pattern falls on the mirror and the reflection is approximately centered on the scale when the beam is in the neutral position (M 's removed). Adjust the distance from the laser to the mirror to obtain a reasonably sharp slit image on the scale. The angular deflection of the light beam is just the ratio of the distance on the scale to the separation between mirror and scale. The angular deflection of the pendulum beam, θ , is just half that amount.

The oscillation frequency ω_1 and damping constant γ are found by fitting measurements of amplitude vs time for free oscillations of the beam to Eq. 8. Oscillations can conveniently be started by placing the large masses in position, waiting about 5 minutes for the beam to reach maximum deflection, and then removing them. The deflection should be noted at regular intervals until the amplitude is too small to easily measure. Fitting can be done with the curve fit option of ASYSTANT or with MATLAB. This determination should probably be repeated 2-3 times in the course of the experiment to check for reproducibility and to establish the uncertainty in the parameters.

Gravitational deflections are determined by placing the large masses in position and waiting several damping times for the beam to come to rest. This procedure will need to be repeated several times for both up and down scale deflections to obtain standard deviations for both directions. The neutral position should also be checked several times. Fortunately the apparatus is sufficiently stable that the measurements can extend over several days if needed.

Once all the data are in hand, you can evaluate G and a probable error. Your report should discuss the relative importance of the various sources of uncertainty, as well as the final result.