Standing Waves

Take but degree away, untune that string, and, hark! What discord follows; each thing meets in mere oppugnancy.

William Shakespeare (1564-1616)

OBJECTIVES

To observe standing waves and resonance in a system with several modes of vibration.

THEORY

In a previous experiment you studied a simple oscillatory system, the RLC circuit. Now we will undertake the study of a more complicated system, a piece of string stretched between two rigid supports. As you will see, the string can support waves and can vibrate at many discrete frequencies, unlike the RLC circuit, which is limited to a single characteristic frequency.

We will initially assume that our string has negligible resistance to bending and neglect all damping effects. If the string has mass per unit length $\sigma$ and the tension is $\tau$, a careful application of Newton's Second Law to a segment of the string leads to the equation of motion

$$\tau \frac{\partial^2 \psi}{\partial x^2} + f(x,t) = \sigma \frac{\partial^2 \psi}{\partial t^2} \tag{1}$$

(The derivation can be found in many mechanics texts.) The function $\psi(x,t)$ describes the displacement of the string at a position $x$ along the string at the time $t$. The time dependent force $f(x,t)$ acts on the string at position $x$.

With a bit of rearrangement, Eq. 1 is recognizable as the familiar wave equation

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = -\frac{f}{\tau} \tag{2}$$

with $v = (\tau/\sigma)^{1/2}$. This equation has many solutions, but we will be interested only in sinusoidal vibrations of the string, which are confined to a single plane, say the vertical. For now we will also set $f = 0$. We can then write one family of solutions as

$$\psi(x,t) = A\cos(\omega t + \phi)\sin \frac{\alpha x}{v} \tag{3}$$
Any point $x$ on the string executes simple harmonic motion in time, and at any instant the shape of the string is given by $\sin(\omega x/v)$. This form of $\psi$ is a solution of the wave equation for any values of $A$ and $\omega$, while $\phi$ is determined by our choice of the instant $t = 0$.

The strings we will deal with are fastened to rigid supports at each end, so the solution (3) must satisfy the boundary conditions

$$\psi(0, t) = \psi(\ell, t) = 0$$

for a string of length $\ell$. Our solution automatically vanishes at $x = 0$, but the second condition requires that $\sin(\omega \ell / v) = 0$, or

$$\omega_n = \frac{n\pi v}{\ell}, \quad n = \text{integer}$$

This result tells us that the string can only vibrate at certain discrete frequencies given by the $\omega_n$. The integer $n$ counts the number of half-periods of the wave that will fit between the ends of the string. This is shown in Fig. 1, for the first few modes. The figure displays "snapshots" of the string at the instant when it is motionless at one extreme position of the motion. One quarter cycle later the string would be straight and moving at maximum velocity. One half cycle later, the string would again be motionless and the relative positions of the parts of the string would be inverted.

Up to this point, we have neglected a number of features of the real situation. The most obvious is damping, which will occur because of the energy losses within the string and because of the energy loss to the sound wave which will be emitted. From our previous work we know that damping will shift the resonant frequency, but only a little if the $Q$ is large. Experimentally $Q \geq 100$, so this effect is small.

We also assumed that the string has negligible resistance to bending. Our experiment will use a steel wire, for which this is not likely to be true. When the wire is bent it tends to spring

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Fig. 1 Standing wave patterns on a string.
back, thereby increasing the restoring force for the vibrations and raising their frequency slightly. The higher modes involve more bending of the wire, so we should expect the effect to become larger as \( n \) increases. (The departure from a harmonic series is important in determining the tone quality of a piano. Electronic keyboard instruments that utilize a perfect harmonic sequence are easily distinguished from a real piano.)

If we include the stiffness we get a wave equation of the form

\[
\tau \frac{\partial^2 \psi}{\partial x^2} - \Gamma \frac{\partial^4 \psi}{\partial x^4} = \sigma \frac{\partial^2 \psi}{\partial t^2}
\]

(6)

where \( \Gamma \) is a constant that depends on the diameter of the wire and the stiffness of the material. Solving this equation analytically is difficult, but if \( \Gamma \) is small the mode frequencies can be written

\[
\omega_n = n\omega_0 \left[ 1 + \beta + \beta^2 + \frac{n^2 \pi^2}{8 \beta^2} \right]
\]

(7)

where \( \omega_0 \) is the fundamental frequency of the same string without stiffness, and \( \beta \) is a constant. When \( \beta = 0 \), as it would be when there is no resistance to bending, this relation is equivalent to Eq. 5. For the wires we will use in the lab, \( \beta \approx 0.007 \), so we will not be able to detect the effect of the first two terms without highly accurate measurements of \( \tau, \sigma \) and \( \ell \). The term in \( n^2 \) can, however, be demonstrated if we make accurate measurements of \( \omega_n \) for large \( n \).

Finally, we should ask what happens if we apply a force \( f \) to the string. Even if the force has a simple sinusoidal time dependence, the mathematics becomes a good deal more complicated. Our previous experience with the RLC circuit, however, suggests that the response to the force will be maximum if the driving frequency is the same as the natural frequency of the string. In this case there are several natural frequencies, so we should expect a resonance whenever \( f(x,t) = f_n \cos(\omega_n t) \), with \( \omega_n \) given by Eq. 7. We can find the resonances by watching (or listening to) the deflection of the string as we slowly adjust the driving frequency.

**EXPERIMENTAL PROCEDURE**

Each station has a box with a steel wire stretched between two supports. The tension in the wire can be adjusted with a thumbscrew at one end. The time-varying force \( f(x,t) \) is provided by centering an electromagnet under the wire and passing a varying current through the magnet. This generates a force on the wire at the same frequency as the driving current.
The wiring diagram is shown in Fig. 2. The function generator is a source of AC voltage, which controls a transistor to drive 1-2 amps of current through the coil. The scope connection allows you to monitor the voltage drop across a resistor in series with the coil and hence the driving current. Power comes from a large DC supply that has been pre-set to the required output.

The motion of the wire is sensed by an optical pickup device. The sensor consists of a light emitting diode opposite a phototransistor light detector. As it moves, the wire blocks more or less of the light, leading to an electrical signal proportional to the deflection for small amplitudes. The optical pick-up should sit on the monocord box with the wire going through the plastic slot on top. The attached electronics box plugs into a small power supply that should be set for full output, 10 V.

Connections to the scope are made from the electronics box and monitor terminal as labeled. Set both channels for x1 probe, and trigger on channel 1, the strong monitor signal. Read the frequency from the trigger display at the bottom of the screen, or use the measure function to separately display frequency from channel 1.

Initially you should make some qualitative observations about the behavior of the system. Adjust the tension so that the fundamental frequency \((n = 1\) mode) is between 100 Hz and 120 Hz. (Pluck the string to hear the pitch. For the musically inclined, this is a note between G\(_2\) and B\(_2\). The rest of us can just guess.) Place the electromagnet near one end of the string, and adjust the function generator to get a large amplitude but undistorted current through the coil. As you slowly vary the frequency you should be able to hear the wire respond and see a change in the output of the optical detector. By tuning very slowly, you should be able to maximize the vibration amplitude of the wire for each of the first two or three modes. Note that if the
electromagnet is placed near a node (a region of zero amplitude) for a given mode, you will not be able to drive that mode. The cure is to move the magnet slightly.

You will probably also notice that the wire does not respond immediately when you change the frequency. This is because the motion of the wire has a transient, or decaying, oscillation at the natural frequencies as well as the steady response at the driving frequency. The overall motion does not become steady until the transient part dies away. Because the $Q$ is very large, the decay time is long enough to be noticeable as you tune the driving frequency.

Once you have located the first few resonances, you should set the function generator to drive the lowest mode, and observe the motion with the strobe lamp. It is probably most informative to adjust the flashing rate so that the wire appears to move slowly. Sketch the motion that you observe, paying particular attention to the relative direction of motion of various parts of the wire. Repeat this procedure for the first three or four modes of the wire. Do your pictures look like Fig. 1?

Next, you should attempt to check Eq. 5. Determine the frequency of the first ten or twelve resonances, and plot the resonance frequency vs $n$. Is Eq. 5 a reasonable approximation to the data?

The effect of wire stiffness, given by Eq. 7, is small but it can be detected if you take sufficient care with the measurements. The fundamental frequency should be somewhat above 100 Hz, and you will have to locate the resonance maxima accurately. With a steady hand you should be able to find the first 20 or 30 modes, although one or two may be missing. It helps to monitor both the sound of the wire and the output of the optical sensor.

Referring to Eq. 7, we see that a plot of $\omega_n/n$ vs $n^2$ should be a straight line with positive slope and intercept near $\omega_1$. If your data fit, it suggests that wire stiffness is the cause of a small shift in harmonic frequencies. If your data do not all fall on a straight line, you may have miscounted a mode number, or mis-measured a frequency.

**REPORT**

Your report should include your mode sketches, with explanations, and the graphs. In addition, you may notice other phenomena in the course of the experiment. A summary of those observations, and perhaps an explanation, would be of interest.