COMP 482 / ELEC 420

Sorting

Your To-Do List

- Read [CLRS] 6-8.
- Assignment 3.

Overview

Should already know various sorting algorithms:

Insertion sort, Selection sort, Quicksort, Mergesort, Heapsort

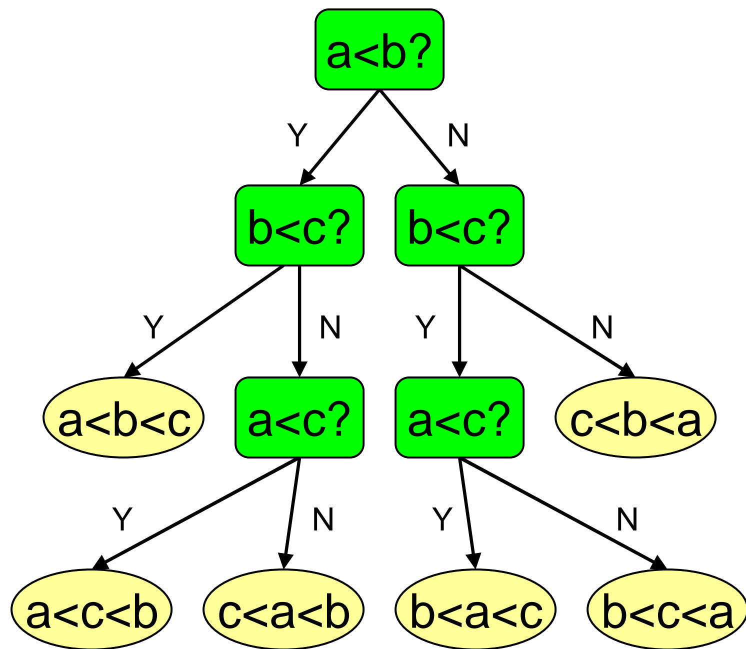
We'll concentrate on ideas not seen in previous courses:

- Lower bounds on general-purpose sorting
- Quicksort probabilistic analysis
- Special-purpose linear-time sorting

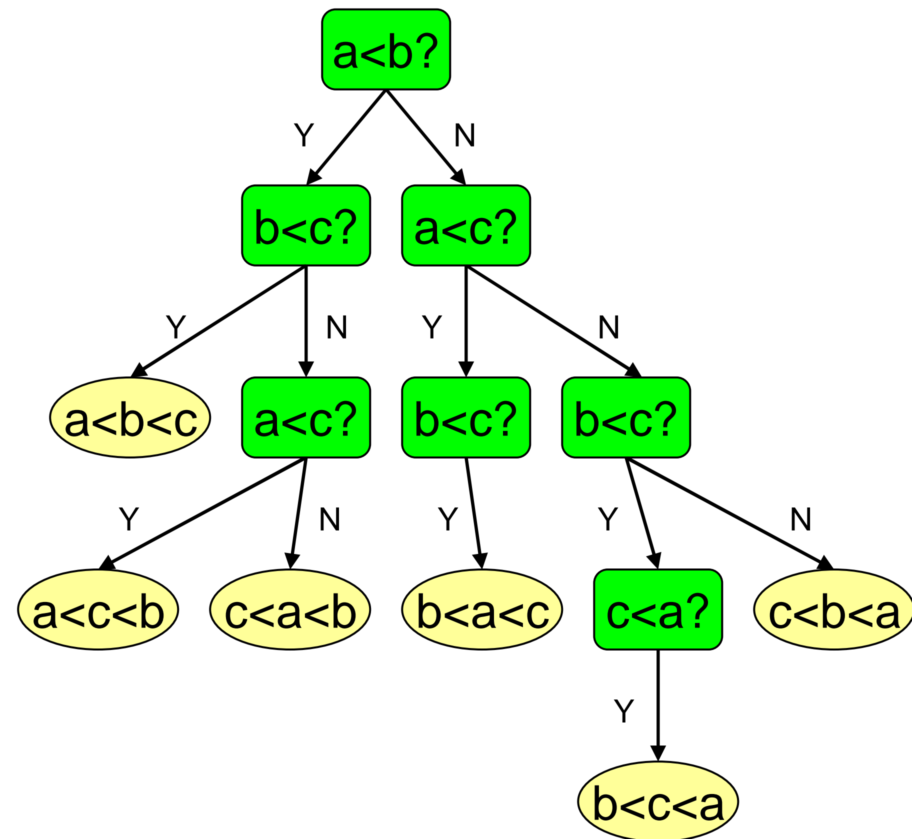
Comparison Trees

Comparisons used to determine order of 3 distinct elements.

Leaves correspond to all possible permutations.



Minimal



Not Minimal

Comparison Trees & Sorting

How does this relate to sorting?

- Any sorting algorithm must be able to reorder any permutation.
- Any sorting algorithm’s behavior corresponds to some comparison tree.

Lower Bounds on Sorting

? How many leaves in a comparison tree? ?

n!

? How many levels? ?

At least lg (n!) = Ω(n lg n).

So, any general sorting algorithm must make Ω(n lg n) comparisons on at least some inputs.

Quicksort

A functional version:

```
qsort(A):
   if |A| = 1
        return A
   else
        pivot = first element of A
        L,G = partition(A, pivot)
        return join(quicksort(L), pivot, quicksort(G))
```

Quicksort

Imperative version more traditional, moving data within the original array.

Details in [CLRS].

Advantage: Lower constant factors in time & space.

Disadvantage: Much more complex.

No asymptotic time or space difference.

Space comparison assumes a slightly more complicated functional version using tail recursion.

Quicksort Analysis: Overview

Should already know: running time of Quicksort depends on a good (lucky?) choice of pivot.

Best case?	Easy analysis
Worst case?	Easy analysis
Average case?	Harder analysis

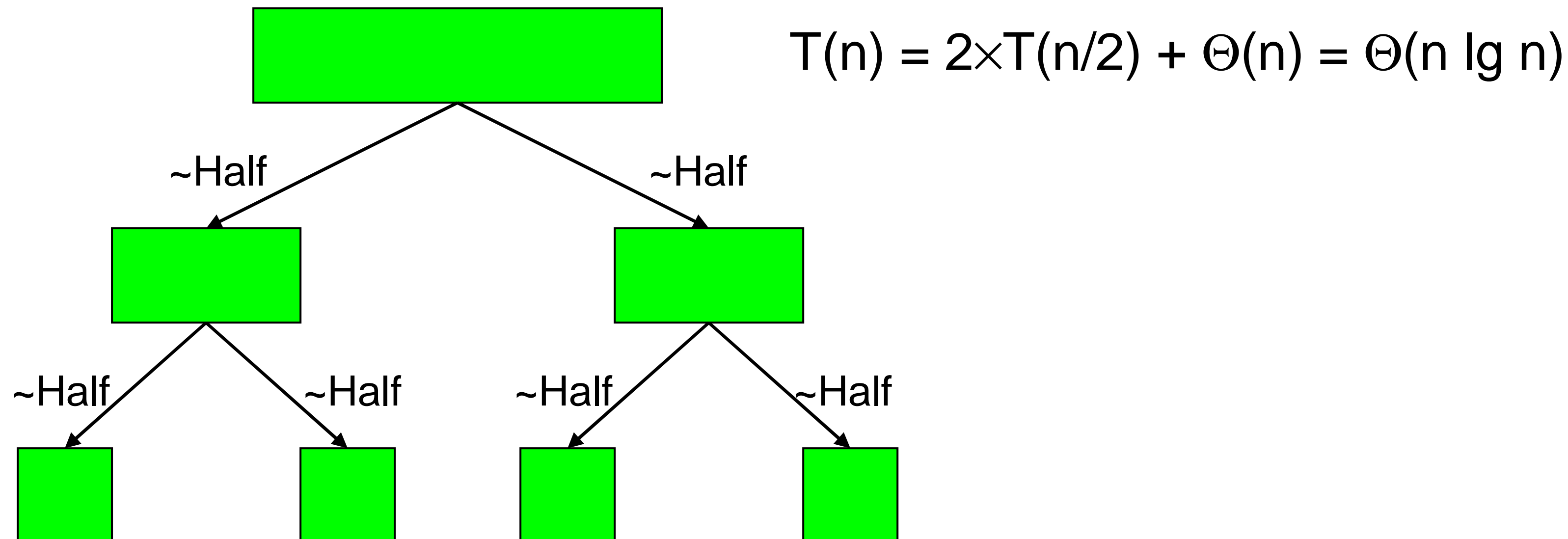
Quicksort Best Case

? When does best case happen? ?

Pivot is always median element.

Again, fairly obvious, but should be proved.

? What is resulting recurrence & bound? ?



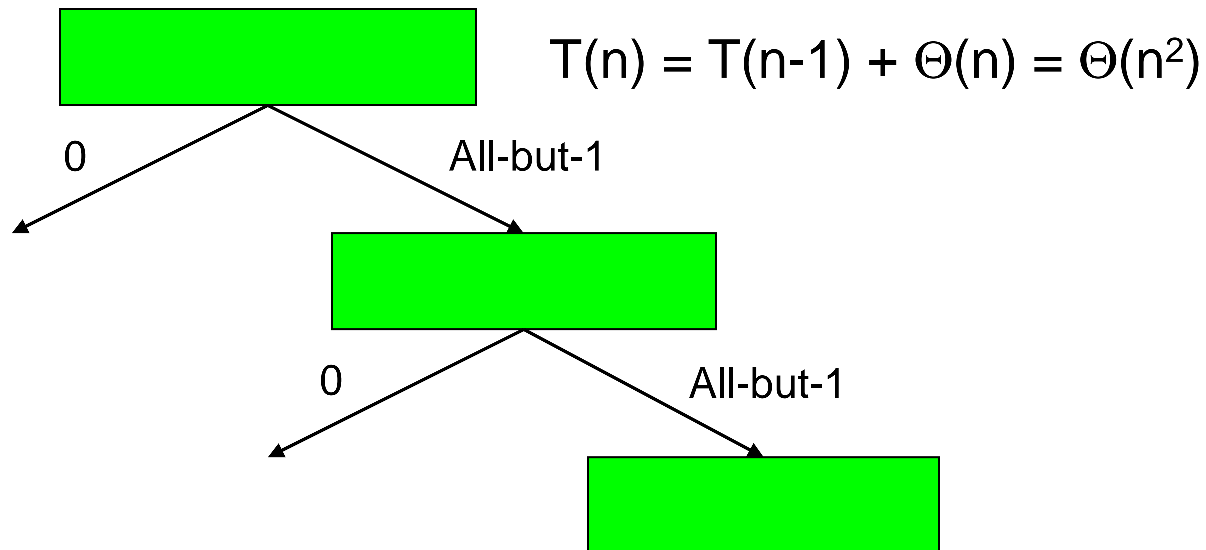
$T(n) = 2\times T(n/2) + \Theta(n) = \Theta(n \lg n)$

Quicksort Worst Case

? When does worst case happen? ?

Pivot is always smallest or largest remaining element.

? What is resulting recurrence & bound? ?



Quicksort Worst Case

Could try different pivot-choosing algorithm.

O(1): Can still be unlucky $\rightarrow$ O(n^2) quicksort worst case

O(n): Can find median (as we'll see soon) $\rightarrow$ O(n log n) quicksort

We took shortcut by assuming the 0 & all-but-1 split is worst.

Intuitively obvious, but could prove this, e.g.,

$T(n) = \max_{q=0..n-1} (T(q) + T(n-q-1)) + \Theta(n)$

...which can be solved to...

$T(n) = \Theta(n^2)$

Quicksort Average Case Overview

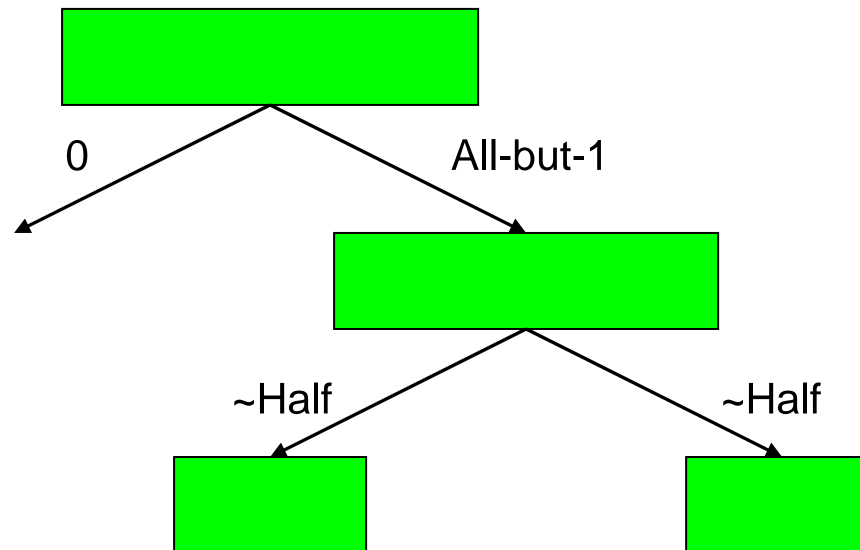
Average case is more like the best case than the worst case.

Two interesting cases for intuition:

1. Any sequence of partitions with the same ratios, such as 1/2::1/2 (the best case), 1/3::2/3, or even 1/100::99/100.
    - As have previously seen, the recursion tree depth is still logarithmic, which leads to the same bound.
    - Thus, the "good" cases don't have to be that good.

Quicksort Average Case Overview

2. Sequence of alternating worst case and best case partitions.
   - Each pair of these partitions behaves like a best case partition, except with higher overhead.



   - Thus, can tolerate having some bad partitions.

Quicksort Average Case Overview

Already have $\Omega(n \log n)$ bound.

Want to obtain $O(n \log n)$.

Can overestimate in analysis.

Always look for ways to simplify!

Quicksort Average Case: Partitioning

Observe: Partitioning dominates Quicksort's work.

- **Partitioning includes the comparisons – the interesting work.**
- Every Quicksort call partitions – except the O(1)-time base cases.
- Partitioning more expensive than joining.

? How many partitions are done in the sort? ?

n-1 = O(n).

Observe: Comparisons dominate partitioning work.

- Each partition's time ∝ that partition's #comparisons.

So, concentrate on time spent comparing.

Quicksort Average Case: Analysis 1

of comparisons in partition for quicksort on n elements

$$C(n) = \ldots$$

# of comps. in this partition? $n-1$

# of comps. in two recursive calls? $C(i)+C(n-i-1)$

Average this over all split positions. $\frac{1}{n}\sum_{i=0}^{n-1}\left(C(i)+C(n-i-1)\right)$

$$C(n) = n-1+\frac{1}{n}\sum_{i=0}^{n-1}\left(C(i)+C(n-i-1)\right)$$

Incorrect formula for average!
Permutations uniformly distributed, but the split positions aren't!

Quicksort Average Case: Analysis 2

Rather than analyzing the time for each partition, and then summing, instead directly analyze the total number of comparisons performed over the whole sort.

Quicksort's behavior depends on only values' ranks, not values themselves.

Z = set of values in array input A.

z_i = i^{th}-ranked value in Z.

Z_{ij} = set of values $\{z_i, \ldots, z_j\}$.

Quicksort Average Case: Analysis 2

Let X_{ij} = I{z_i is compared to z_j}

Total comparisons $$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

Each pivot is selected at most once, so each z_i,z_j pair is compared at most once.

$$E[X] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$ by linearity of expectation

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared to } z_j\}$$ by definition

Quicksort Average Case: Analysis 2

What is this probability?

Consider arbitrary i,j and corresponding Z_{ij}.

Z_{ij} need not correspond to a partition executed during the sort.

Claim: z_i and z_j are compared $\leftrightarrow$ either is the first element in Z_{ij} to be chosen as a pivot.

Proof: Which is first element in Z_{ij} to be chosen as pivot?

- If z_i, then that partition must start with at least all the elements in Z_{ij}. Then z_i compared with all the elements in that partition (except itself), including z_j.
- If z_j, similar argument.
- If something else, the resulting partition puts z_i and z_j into separate sets (without comparing them), so that no future Quicksort or partition call will consider both of them.

Quicksort Average Case: Analysis 2

Now, compute the probability:

$Pr\{z_i$ is compared to $z_j\}$

= $Pr\{z_i$ or z_j is first pivot chosen from $Z_{ij}\}$ — just explained

= $Pr\{z_i$ is first pivot chosen from $Z_{ij}\}$ +
$Pr\{z_j$ is first pivot chosen from $Z_{ij}\}$ — mutually exclusive possibilities

= $1/(j-i+1) + 1/(j-i+1)$

= $2/(j-i+1)$

Quicksort Average Case: Analysis 2

Plug this back into the sum:

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n-1} \sum_{k=2}^{n-i-1} \frac{2}{k}$$

Simplify with a change of variable, $k = j - i + 1$.

$$< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k}$$

Simplify and overestimate, by adding terms.

$$= \sum_{i=1}^{n-1} O(\log n)$$

$$= O(n \log n)$$

Quicksort Analysis

? Are all inputs equally likely in practice? ?

Often, no.

E.g., in many situations, data is “almost sorted”, which leads to more bad partitions.

? How can we avoid this? ?

Randomize the input.

Linear-Time Sorting

In limited circumstances, can avoid comparison-based sorting, & thus do better than previous lower bound!

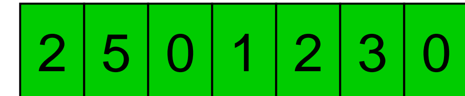
Must rely on some restriction on inputs.

Counting Sort

Limit data to a small discrete range: e.g., 0,…,5.

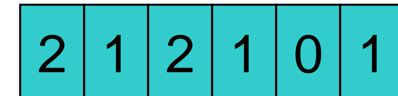
Let m = size of range.

Input=



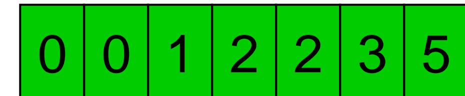
Count instances of each possible element:

Count=



Produce Count[i] instances of each i:

Output=



Limited usefulness, but the simplest example of a non-comparison-based sorting algorithm.

Counting Sort

```
csort(A,n):
    /* Count number of instances of each possible element. */
    Count[0…m-1] = 0                                  Θ(m)
    For index = 0 to n-1
        Count[A[index]] += 1                          Θ(n)

    /* Produce Count[i] copies of each i. */
    index = 0                                    ⎫
    For i = 0 to m-1                             ⎪
        For copies = 0 to Count[A[i]]            ⎬  Θ(m+n)
            A[index] = i                         ⎪
            index += 1                           ⎭
```

<u>$\Theta(m+n) = \Theta(n)$ time</u>, when m taken to be a constant.

Bucket Sort

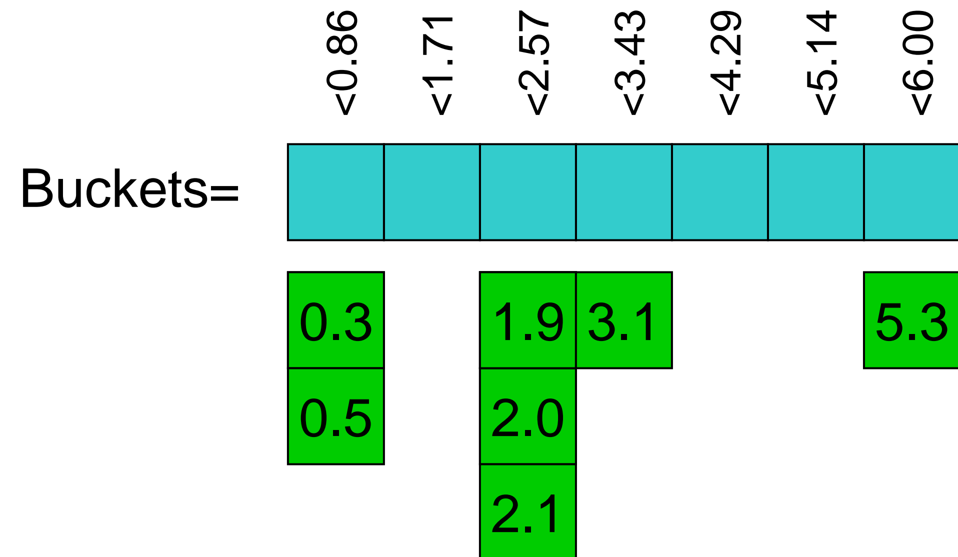
Limit data to a continuous range: e.g., [0,6).

Let m = size of range.

Input=

2.1	5.3	0.5	1.9	2.0	3.1	0.3

Create n buckets, for equal-sized subranges



For each

- Calculate bucket = $\lfloor d \cdot n/m \rfloor$
- Insert to bucket's list.

$2.1 \rightarrow$ Bucket $\lfloor 2.1 \cdot 7/6 \rfloor = \lfloor 2.45 \rfloor = 2$

Bucket Sort Analysis

	Worst Case	Best Case	Average Case
Items per Bucket	n	1	O(1)
Total Time	$O(n^2)$	O(n)	O(n)

Radix Sort

- Limit input to fixed-length numbers or words.
  - Represent symbols in some base b.
  - Each input has exactly d “digits”.

- Sort numbers d times, using 1 digit as key.
  - Must sort from least-significant to most-significant digit.
  - Must use any “stable” sort, keeping equal-keyed items in same order.

Radix Sort Example

Input data:

a	b	a

b	a	c

c	a	a

a	c	b

b	a	b

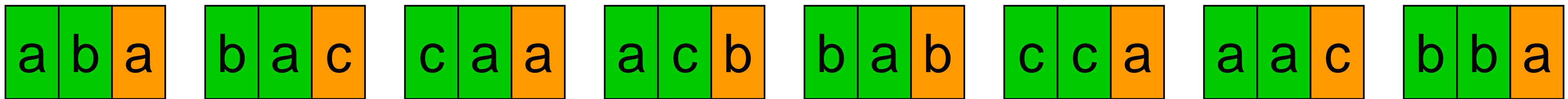
c	c	a

a	a	c

b	b	a

Radix Sort Example

Pass 1: Looking at rightmost position.

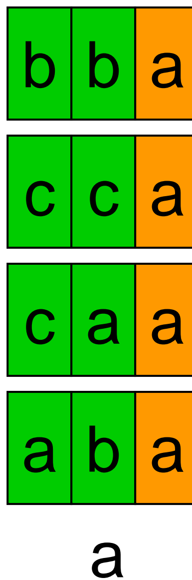


Place into appropriate pile.

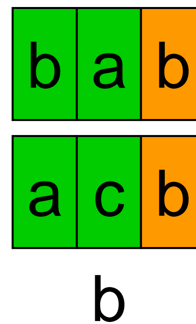
a b c

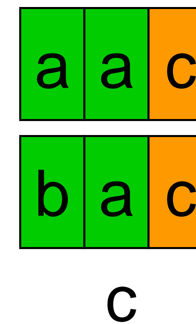
Radix Sort Example

Pass 1: Looking at rightmost position.



Join piles.

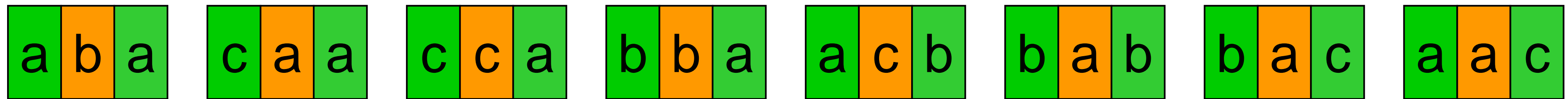




a b c

Radix Sort Example

Pass 2: Looking at next position.

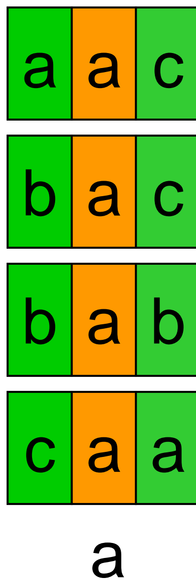


Place into appropriate pile.

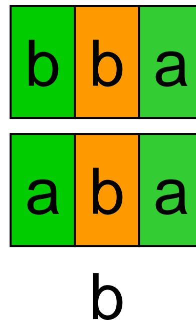
a b c

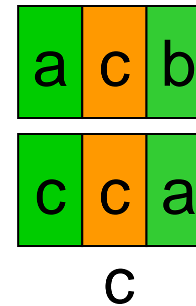
Radix Sort Example

Pass 2: Looking at next position.



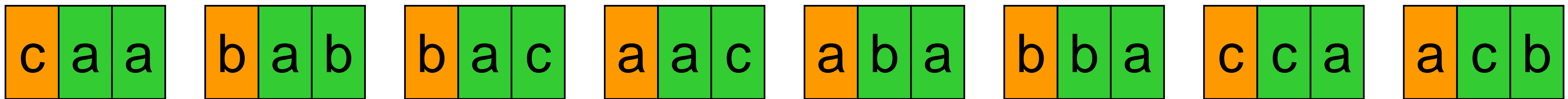
Join piles.





Radix Sort Example

Pass 3: Looking at last position.



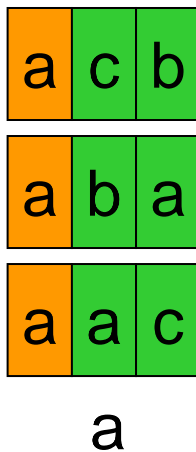
Place into appropriate pile.

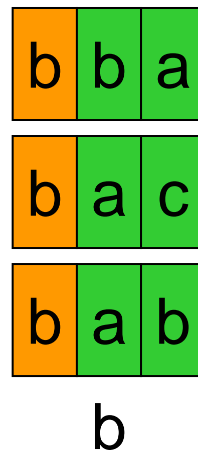
a b c

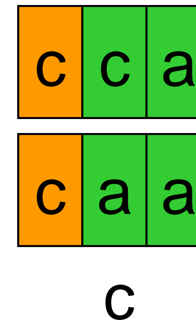
Radix Sort Example

Pass 3: Looking at last position.

Join piles.







a b c

Radix Sort Example

Result is sorted.

a	a	c

a	b	a

a	c	b

b	a	b

b	a	c

b	b	a

c	a	a

c	c	a

Radix Sort Algorithm

```
rsort(A,n):
    For j = 0 to d-1
        /* Stable sort A, using digit position j as the key. */
        For i = 0 to n-1
            Add A[i] to end of list ((A[i]>>j) mod b)

        A = Join lists 0…b-1
```

$\Theta(dn)$ time, where d is taken to be a constant.

Some Applets

Counting, Bucket, & Radix Sort

http://algoviz.cs.vt.edu/AlgovizWiki/RadixSort