<u>COMP 360 -- Midterm Exam (200 Points)</u>

Below are the rules for this examination:

- a <u>four</u> hour time limit to finish the entire exam;

- you need not do the entire exam in one session, but once you read a problem you must finish the problem without taking time out (except for obvious emergencies);

- Please write all your answers in **INK**, not in pencil! Pencil is difficult to read. If you want to change an answer, just cross out your old answer and write your new answer. Do not try to erase.

- <u>closed</u> textbook, <u>closed</u> notes; some helpful identities appear on pages 2,3.

- all work must be entirely your own;

- you may not confer on any problems with any person except the professor;

- the exam must be handed in to no later than the start of class on Tuesday, October 13.

There are 5 problems on the exam. Each problem is worth 40 points. For full credit you must do **ALL** 5 problems.

Good luck.

Dot Product

$u \bullet v = \lvert u \rvert \, \lvert v \rvert \cos(\theta)$	(definition)
$u \bullet v = v \bullet u$	(commutative)
$u \bullet (v + w) = u \bullet v + u \bullet w$ $(v + w) \bullet u = v \bullet u + w \bullet u$	(distributive)
$\lvert v \rvert^2 = v \bullet v$	(length)
$\cos(\theta) = \dfrac{u \bullet v}{\lvert u \rvert \, \lvert v \rvert}$	(angle)
$u_{\parallel} = \left(\dfrac{u \bullet v}{v \bullet v} \right) v$ $u_{\parallel} = (u \bullet v) v \quad \textit{if} \quad \lvert v \rvert = 1$	(parallel projection)
$u_{\perp} = u - u_{\parallel} = u - \left(\dfrac{u \bullet v}{v \bullet v} \right) v$ $u_{\perp} = u - (u \bullet v) v \quad \textit{if} \quad \lvert v \rvert = 1$	(perpendicular projection)
$u \bullet v = 0 \Leftrightarrow u \perp v$	(orthogonality)

Cross Product

$\lvert u \times v \rvert = \lvert u \rvert \lvert v \rvert \sin(\theta)$ $u \times v \perp u, v$ $\operatorname{sgn}(u, v, u \times v) > 0$	(definition)
$area(u, v) = \lvert u \times v \rvert$	(area)
$(u \times v) \bullet u = 0$ $(u \times v) \bullet v = 0$	(orthogonality)
$u \times u = 0$ $u \times v = 0 \Leftrightarrow v \parallel \pm u$	(parallelism)
$u \times (v + w) = u \times v + u \times w$ $(v + w) \times u = v \times u + w \times u$	(distributive)
$u \times v = -v \times u$	(anti-commutative)
$u \times (v \times w) = (u \bullet w) v - (u \bullet v) w$ $(v \times w) \times u = (u \bullet v) w - (u \bullet w) v$	(non-associative)
$\lvert u \times v \rvert^2 = \lvert u \rvert^2 \lvert v \rvert^2 - (u \bullet v)^2$	(length)
$(u_1 \times u_2) \bullet (v_1 \times v_2) = (u_1 \bullet v_1)(u_2 \bullet v_2) - (u_1 \bullet v_2)(u_2 \bullet v_1)$	(Lagrange Identity)

Determinant

$\det(u,v,w) = (u \times v) \bullet w$ (definition)

$vol(u,v,w) = |\det(u,v,w)|$ (volume)

$\det(u,v,w) > 0 \Leftrightarrow \operatorname{sgn}(u,v,w) > 0$ (orientation)

$\det(u,v,w) \neq 0 \Leftrightarrow u,v,w \ \ \textit{are linearly independent}$
$\det(u,u,w) = \det(u,v,u) = \det(u,v,v) = 0$ (linear independence)

$\det(u,v,w) = \det(v,w,u) = \det(w,u,v)$
$\det(v,u,w) = -\det(u,v,w)$ (skew symmetry)

$\det(u_1 + c\,u_2, v, w) = \det(u_1, v, w) + c\det(u_2, v, w)$
$\det(u, v_1 + c\,v_2, v, w) = \det(u, v_1, w) + c\det(u, v_2, w)$ (multi-linearity)
$\det(u, v, w_1 + c\,w_2) = \det(u, v, w_1) + c\det(u, v, w_2)$

1. (40 points)

Consider the hexagonal bump depicted in Figure 1:

a. Write a turtle program to generate this bump.

b. Write a turtle program to create the bump fractal in Figure 2 generated by the bump in Figure 1.

c. Describe mathematically the transformations in the iterated function system (IFS) that generate the fractal in Figure 2.

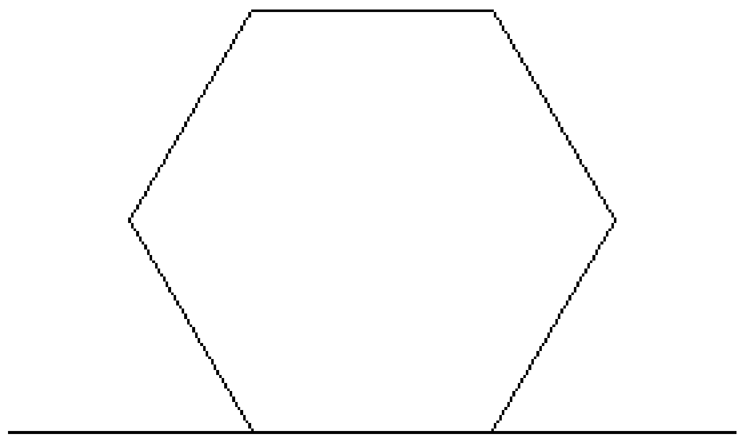

Figure 1: A hexagonal bump. Start at the left end point of the line at the base of the figure, proceed counterclockwise around the hexagon, and finish at the right end point of the line.

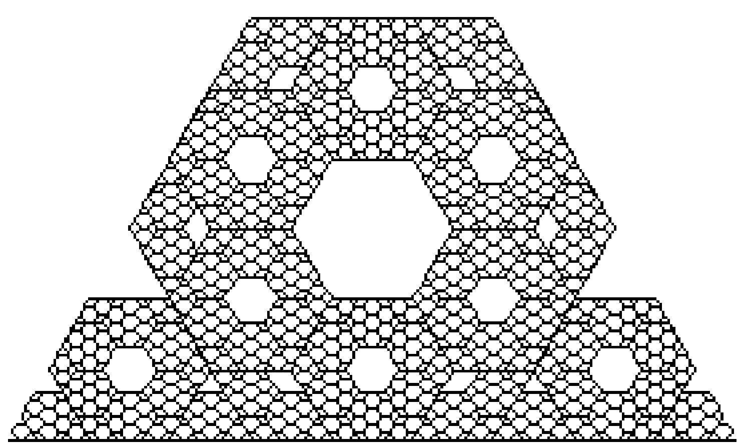

Figure 2: The bump fractal generated by the bump in Figure 2.

2. (40 points)

Consider the fractal flower depicted in Figure 3:

a. Write a turtle program to generate this fractal flower.

b. Describe the transformations in the iterated function system (IFS) that generate this fractal flower.

c. Compute the fractal dimension of this fractal flower.

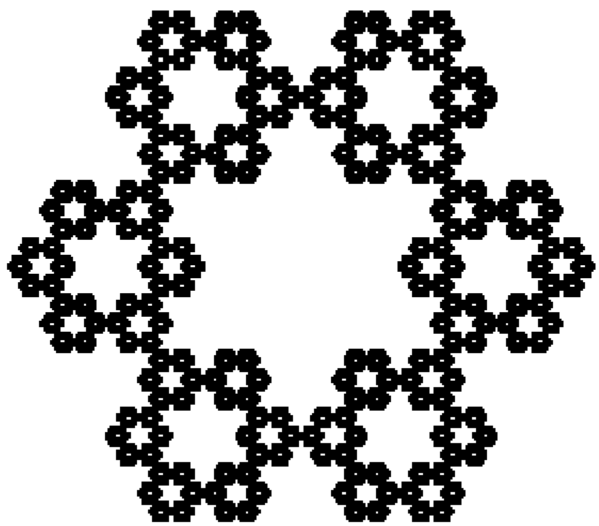

Figure 3: A fractal flower.

3. (40 points)

a. Show that in 2-dimensions every conformal linear transformation is uniquely determined by either of the following:

  i. the image of one point and one vector.

  ii. the image of two points.

b. Let M, N be two 3×3 matrices whose third column is $(0 \quad 0 \quad 1)^T$.

Show that:

  i. the third column of $M * N$ is $(0 \quad 0 \quad 1)^T$;

  ii. the third column of M^{-1} is $(0 \quad 0 \quad 1)^T$.

c. Let M be a 3×3 matrix, and let v,w be arbitrary vectors. Show that

  i. $(v * M) \bullet w = v \bullet (w * M^T)$

  ii. $(u * M) \times (v * M) \neq (u \times v) * M$

d. Let w_1, w_2, w_3 be the images of the three linearly independent vectors v_1, v_2, v_3 under the affine transformation A. Show that if u is an arbitrary vector, then

$$A(u) = \frac{Det(u \;\; v_2 \;\; v_3)\, w_1 + Det(v_1 \;\; u \;\; v_3)\, w_2 + Det(v_1 \;\; v_2 \;\; u)\, w_3}{Det(v_1 \;\; v_2 \;\; v_3)}.$$

(Hint: Use the properties of determinants.)

4. (40 points)

a. Without appealing to coordinates, prove that

$$|u \times v|^2 = |u|^2 |v|^2 - (u \bullet v)^2.$$

b. Let u, w be unit vectors and define the map $w \otimes u$ by setting

$$(w \otimes u)(v) = (v \bullet w)u \ .$$

Show that:

i. $w \otimes u$ is a linear transformation

ii. $(u \otimes u)(v) = v_{||}$, where $v_{||}$ is the parallel projection of v on u.

c. Let v_0, v_1 be unit vectors. Show that the unit vector that bisects the angle between v_0 and v_1 is

$$v = \sec(\phi/2)\left(\frac{v_0 + v_1}{2}\right),$$

where ϕ is the angle between v_0 and v_1.

d. Consider a line L through a point P in the direction v, and a plane S through a point Q with normal vector N. Show that the line L and the plane S intersect in the point R, where

$$R = P + \frac{N \bullet (Q - P)}{N \bullet v} v.$$

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5. (40 points)

Parallel projection is projection onto a plane S along the direction of a unit vector u. As usual the plane S is defined by a point Q on S and a unit vector n normal to S (see Figure 4).

a. Show that under parallel projection for any vector v and any point P

i. $v^{new} = v - \frac{v \bullet n}{u \bullet n} u$

ii. $P^{new} = P - \frac{(P - Q) \bullet n}{u \bullet n} u$

b. Show that the 4×4 matrix representing parallel projection is given by

$$PProj(Q,n,u) = \begin{pmatrix} I - \frac{n^T * u}{u \bullet n} & 0 \\ \frac{Q \bullet n}{u \bullet n} u & 1 \end{pmatrix}.$$

c. Suppose that

- S is the *xy-plane*,
- u is the unit vector parallel to the vector $(1,1,1)$,

Write down the explicit entries of the 4×4 matrix $PProj(Q,n,u)$.

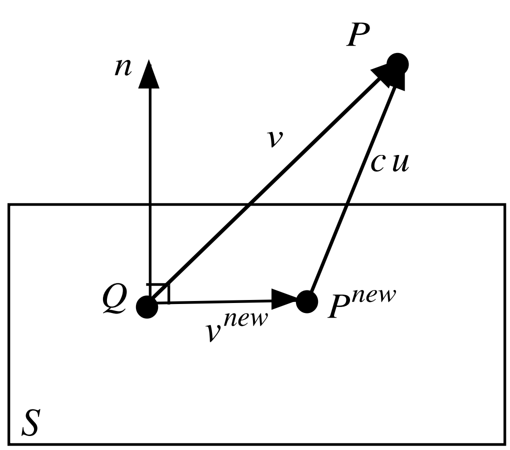

Figure 4: Parallel projection along a direction u into a plane S.