<u>COMP 360 -- Midterm Exam (200 Points)</u>

The following are the rules for this examination:

- a three hour time limit to finish the entire exam;
- you need not do the entire exam in one session, but once you read a problem you must finish the problem without taking time out (except for obvious emergencies);
- Please write all your answers in **INK**, not in pencil! Pencil is difficult to read. If you want to change an answer, just cross out your old answer and write your new answer. Do not try to erase.
- <u>closed</u> textbook, <u>closed</u> notes; some helpful identities appear on pages 2,3.
- all work must be entirely your own;
- you may not confer on any problems with any person except the professor;
- the exam must be handed in to no later than the start of class on Tuesday, October 21.

There are 5 problems on the exam. Each problem is worth 40 points. For full credit you must do **ALL** 5 problems.

Good luck.

Dot Product

$u \bullet v = |u|\,|v| \cos(\theta)$ (definition)

$u \bullet v = v \bullet u$ (commutative)

$u \bullet (v + w) = u \bullet v + u \bullet w$

$(v + w) \bullet u = v \bullet u + w \bullet u$ (distributive)

$|v|^2 = v \bullet v$ (length)

$\cos(\theta) = \dfrac{u \bullet v}{|u|\,|v|}$ (angle)

$u_{\|} = \left(\dfrac{u \bullet v}{v \bullet v}\right) v$

$u_{\|} = (u \bullet v) v \quad if \quad |v| = 1$ (parallel projection)

$u_{\perp} = u - u_{\|} = u - \left(\dfrac{u \bullet v}{v \bullet v}\right) v$

$u_{\perp} = u - (u \bullet v) v \quad if \quad |v| = 1$ (perpendicular projection)

$u \bullet v = 0 \Leftrightarrow u \perp v$ (orthogonality)

Cross Product

$|u \times v| = |u|\,|v| \sin(\theta)$

$u \times v \perp u, v$ (definition)

$\mathrm{sgn}(u, v, u \times v) > 0$

$area(u, v) = |u \times v|$ (area)

$(u \times v) \bullet u = 0$

$(u \times v) \bullet v = 0$ (orthogonality)

$u \times u = 0$

$u \times v = 0 \Leftrightarrow v \,\|\, \pm u$ (parallelism)

$u \times (v + w) = u \times v + u \times w$

$(v + w) \times u = v \times u + w \times u$ (distributive)

$u \times v = -v \times u$ (anti-commutative)

$u \times (v \times w) = (u \bullet w) v - (u \bullet v) w$

$(v \times w) \times u = (u \bullet v) w - (u \bullet w) v$ (non-associative)

$|u \times v|^2 = |u|^2 |v|^2 - (u \bullet v)^2$ (length)

$(u_1 \times u_2) \bullet (v_1 \times v_2) = (u_1 \bullet v_1)(u_2 \bullet v_2) - (u_1 \bullet v_2)(u_2 \bullet v_1)$ (Lagrange Identity)

Determinant

$\det(u,v,w) = (u \times v) \bullet w$ (definition)

$vol(u,v,w) = |\det(u,v,w)|$ (volume)

$\det(u,v,w) > 0 \Leftrightarrow \mathrm{sgn}(u,v,w) > 0$ (orientation)

(linear independence)

$\det(u,v,w) \neq 0 \Leftrightarrow u,v,w \;\; \textit{are linearly independent}$

$\det(u,u,w) = \det(u,v,u) = \det(u,v,v) = 0$

(skew symmetry)

$\det(u,v,w) = \det(v,w,u) = \det(w,u,v)$

$\det(v,u,w) = -\det(u,v,w)$

(multi-linearity)

$\det(u_1 + c\,u_2, v, w) = \det(u_1, v, w) + c\det(u_2, v, w)$

$\det(u, v_1 + c\,v_2, v, w) = \det(u, v_1, w) + c\det(u, v_2, w)$

$\det(u, v, w_1 + c\,w_2) = \det(u, v, w_1) + c\det(u, v, w_2)$

1. (40 points)

   Consider the rectangular bump and the corresponding bump fractal depicted in Figure 1:

   a. Write a turtle program to generate the rectangular bump.

   b. Write a turtle program to create the corresponding bump fractal generated by this bump.

   c. How many transformations would you need to generate this bump fractal using an iterated function system (IFS)?

   d. Describe the transformations in the iterated function system (IFS) that generate this bump fractal.

   e. Compute the fractal dimension of this bump fractal.

Figure 1: A rectangular bump and the corresponding bump fractal.

2. (40 points)

Consider the fractal tree depicted in Figure 2:

a. Write a turtle program to generate this fractal tree.

b. Describe the transformations in the iterated function system (IFS) that generate this fractal tree.

c. Describe the condensation set of this fractal tree.

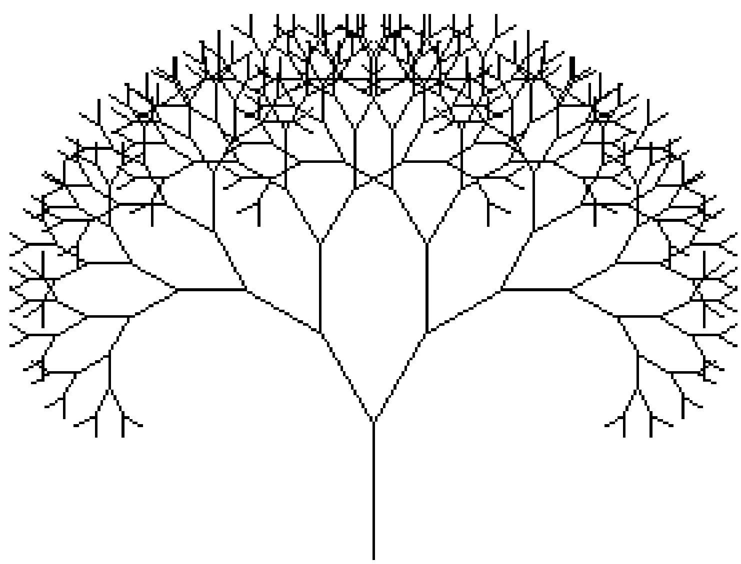

Figure 2: A fractal tree.

3. (40 points)

a. Show that in 2-dimensions an affine transformation is uniquely determined by the image of two points and one vector that is not parallel to the line determined by the two points.

b. Let P_1^*, P_2^*, v^* be the images of P_1, P_2, v under the affine transformation A. Describe the 3×3 matrix M that represents the affine transformation A.

c. Show that in 3-dimensions an affine transformation A is uniquely determined by the image of one point P and three linearly independent vectors v_1, v_2, v_3.

d. Let w_1, w_2, w_3 be the images of the three linearly independent vectors v_1, v_2, v_3 under the affine transformation A. Show that if u is an arbitrary vector, then

$$A(u) = \frac{Det(u \;\; v_2 \;\; v_3)\, w_1 + Det(v_1 \;\; u \;\; v_3)\, w_2 + Det(v_1 \;\; v_2 \;\; u)\, w_3}{Det(v_1 \;\; v_2 \;\; v_3)}.$$

(Hint: Use the properties of determinants.)

4. (40 points)

   a. Without appealing to coordinates, prove that

$$|u \times v|^2 = |u|^2 |v|^2 - (u \bullet v)^2.$$

   b. Let P,Q,R be points in the xy-plane. Prove that

$$Area(\Delta PQR) = \frac{1}{2} Det\begin{pmatrix} P & 1 \\ Q & 1 \\ R & 1 \end{pmatrix} = \frac{1}{2} Det\begin{pmatrix} p_1 & p_2 & 1 \\ q_1 & q_2 & 1 \\ r_1 & r_2 & 1 \end{pmatrix}.$$

   c. Let v_0, v_1 be unit vectors. Show that the unit vector that bisects the angle between v_0 and v_1 is

$$v = \sec(\phi/2)\left(\frac{v_0 + v_1}{2}\right),$$

   where ϕ is the angle between v_0 and v_1.

   d. By drawing the appropriate figures, show that:

      i. $P + (Q - P) = Q$

      ii. $(R - Q) + (Q - P) = R - P$

      iii. $(P + v) + w = P + (v + w)$

      iv. $u + (-1)v = u - v$

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5. (40 points) A *shear transformation* is defined in terms of a shearing plane S, a unit vector u in the plane S, and an angle ϕ in the following fashion. Given any point P, project P orthogonally onto a point P' in the shearing plane S. Now slide P parallel to u to a point P^{new} so that $\angle P^{new} P' P = \phi$. The point P^{new} is the result of applying the shearing transformation to the point P (see Figure 3).

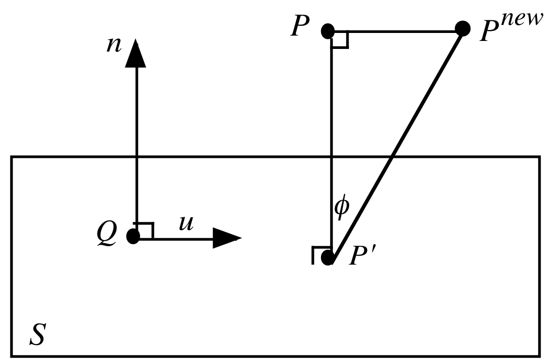

Figure 3: Shear

Let

S = Shearing plane

n = Unit vector perpendicular to S

Q = Point on S

u = Unit vector in S (i.e. unit vector perpendicular to n)

ϕ = Shear angle

a. Show that for any point P

$$P^{new} = P + \tan(\phi)\big((P - Q) \bullet n\big)u$$

b. Show that the 4×4 matrix representing shear is given by

$$Shear(Q,n,u,\phi) = \begin{pmatrix} I + (tan\phi)(n^T * u) & 0 \\ -tan\phi(Q \bullet n)u & 1 \end{pmatrix}.$$

c. Suppose that

i. S is the xy-plane,

ii. u is the unit vector along the x-axis,

iii. $\phi = 45^\circ$

Write down the explicit entries of the 4×4 matrix $Shear(Q,n,u,\phi)$.