Combinatorics

Themes and Applications

Themes

- Counting
- Probability

Applications

- Analysis of Algorithms
  - -- Complexity
  - -- (Average) Speed and Size

Basic Counting Rules

Basic Counting Rules

Sum Rules -- OR

- $|A \cup B| = |A| + |B| \qquad A \cap B = \phi$
- $|A \cup B| = |A| + |B| - |A \cap B|$

Product Rules -- AND

- $|A \times B| = |A|\,|B|$
- $|A_1 \times A_2 \times \cdots \times A_n| = |A_1|\,|A_2| \cdots |A_n|$

Complement Rule -- NOT

- $A \subset S \Rightarrow |A| = |S| - |A^c|$

<u>**Sum Rule**</u>

Rule

- $|A \cup B| = |A| + |B| - |A \cap B|$

Examples

- How many integers between 1 and 1500 are divisible by either 3 <u>OR</u> 5.
  - -- Divisible by 3 = 1500/3 = 500
  - -- Divisible by 5 = 1500/5 = 300
  - -- Divisible by both = 1500/15 = 100
  - -- Total = 500 + 300 – 100 = 700

- Choice of courses
  - -- # math courses = 11
  - -- # computer science courses = 13
  - -- # total choices of math <u>OR</u> computer science = 24
  - -- 3 cross listed $\Rightarrow$ # total choices = 24–3 = 21

<u>**Product Rule**</u>

Rule

$|A_1 \times A_2 \times \cdots \times A_n| = |A_1||A_2|\cdots|A_n|$

Examples

- Passwords: $aaa\ \ xxx$ -- $(26^3)(10^3) = 17{,}576{,}000$
- $|P(A)| = 2^{|A|}$ -- each element is either in or out of the subset
- Functions $f : A \to B$, where $|A| = m$ and $|B| = n$

  -- # function = n^m

  -- # 1-1 functions = $n(n-1)\cdots(n-m+1)$ $\qquad m \le n$

  -- # onto functions = later $\qquad m \ge n$

Product Rule (continued)

Rule

- $|A_1 \times A_2 \times \cdots \times A_n| = |A_1||A_2|\cdots|A_n|$

More Examples

- Boolean Functions: $f: \{0,1\}^n \to \{0,1\}$ -- 2^{2^n} such maps
  - -- 2^n arguments and 2 possible outcomes
  - -- $n = 2 \Rightarrow 2^{2^n} = 16$
  - -- The usual operations $\wedge, \vee, \to$, etc. = Truth Tables

- Relations $R \subset A \times B$, where $|A| = m$ and $|B| = n$
  - -- $P(A \times B) \supset R$
  - -- # relations = $|P(A \times B)| = 2^{|A|\,|B|} = 2^{mn}$
  - -- # equivalence relations on A -- Extra Credit (Section 8.5, #68)

Product Rule (continued)

Rule

- $|A_1 \times A_2 \times \cdots \times A_n| = |A_1||A_2|\cdots|A_n|$

Nested Loops

- For $j_1 = 1 \ to \ p_1$

  For $j_2 = 1 \ to \ p_2$

  $\ddots$

  For $j_n = 1 \ to \ p_n$

  Execute Some Actions

- Total Number of Times Through Loops = $p_1 \cdots p_n$

<u>Product Rule (continued)</u>

Rule

- $|A_1 \times A_2 \times \cdots \times A_n| = |A_1||A_2|\cdots|A_n|$

More Examples

- # of bit strings of length $\leq n, \quad n \geq 1$

  -- # of bit strings of length $k = 2^k$ $\qquad 1 \leq k \leq n$

  -- Total $= 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 2$

- Choice of Courses

  -- # math courses = 11

  -- # computer science courses = 13

  -- # total choices of math <u>and</u> computer science = $11 \times 24 = 264$

- Number of Fraternities with 3 Greek Letters

  -- Greek has 24 letters

  -- # fraternity names = $24 * 24 * 24 = 13,824$

Complement Rule

Rule

- $A \subset S \Rightarrow |A^c| = |S| - |A|$

Examples

- How many Integers between 1 and 1500 are NOT divisible by 3?

  -- Divisible by 3 = 1500/3 = 500

  -- Not Divisible by 3 = 1500 – 500 = 1000

- How many Passwords: *aaa xxx* NOT beginning with *A* and ending in 0?

  -- Total Number of Passwords = $(26^3)(10^3) = 17{,}576{,}000$

  --- Passwords beginning with *A* and ending with 0 = $(26^2)(10^2) = 67{,}600$

  -- NOT beginning with *A* and ending in 0 = $17{,}576{,}000 - 67{,}600 = 17{,}508{,}400$

Pigeonhole Principles

Rules

- If you put n objects into k boxes, $n > k$, then at least one of the boxes must contain at least 2 elements.
- If you put n objects into k boxes, $n > pk$, then at least one of the boxes must contain at least $p+1$ elements.

Proof

- By Contradiction

Consequences

- If $f : A \rightarrow B$ and $|A| > |B|$, then f is not 1-1.
- If $f : A \rightarrow B$ and $|A| < |B|$, then f is not onto.

Pigeonhole Principle (continued)

Rule

- If you put n objects into k boxes, $n > p\,k$, then at least one of the boxes must contain at least $p+1$ elements.

Chromatic Triangles

- In the Complete Graph with 6 Vertices and 15 Lines (K_6) Color Each Line Red or Blue
- Conclusion: At Least One Triangle has all its Edges the Same Color

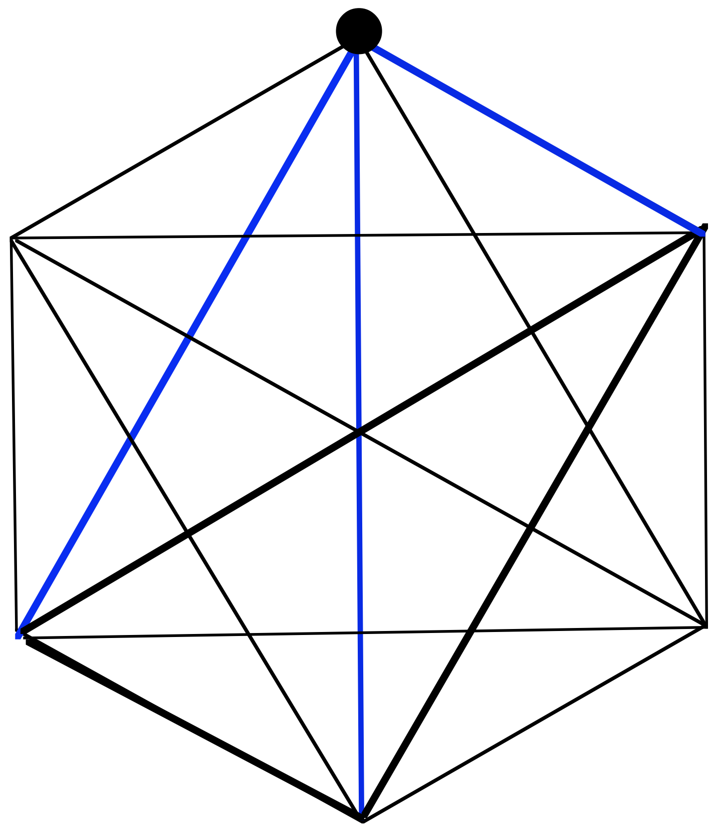
Proof: Pick any vertex. At least three edges from this vertex have the same color -- say blue (pigeonhole principle).

Consider the triangle made from the vertices on these lines opposite to A.

Either these edges all have the same color (red) or one of them is blue,

in either case establishing the result.

Chromatic Triangle



K_6: The Complete Graph with 6 Vertices and 15 Edges

Pigeonhole Principle (continued)

Rule

- If you put n objects into k boxes, $n > k$, then at least one of the boxes must contain at least 2 elements.

Eternal Return

- Eventually everything you experience will happen again.
  - The universe is finite.
  - The universe is deterministic.
  - The exact state of the universe will recur in a finite time.
- Computer States
  - A computer with m memory locations that performs one operation every *nth* of a second will necessarily have been in the same state at least twice after m/n seconds.

Pigeonhole Principle (continued)

Rule

- If you put n objects into k boxes, $n > p\,k$, then at least one of the boxes must contain at least $p+1$ elements.

Theorem: Let $|A| = n$. *If there is a path in R from a to b, then there is a path in R from a to b of length at most n* ($n-1$ *if* $a \neq b$).

Proof: Remove cycles. Pigeonhole Principle.

Corollary: $|A| = n \Rightarrow R^{*} = R \cup R^{2} \cup \cdots \cup R^{n}$

Summary: Basic Counting Principles

Sum Rules -- OR

- $|A \cup B| = |A| + |B| \qquad A \cap B = \phi$
- $|A \cup B| = |A| + |B| - |A \cap B|$

Product Rules -- AND

- $|A \times B| = |A|\,|B|$
- $|A_1 \times A_2 \times \cdots \times A_n| = |A_1|\,|A_2| \cdots |A_n|$

Complement Rule -- NOT

- $A \subset S \Rightarrow |A| = |S| - |A^c|$

<u>Summary: Pigeonhole Principles</u>

First Pigeonhole Principle

- If you put n objects into k boxes, $n > k$, then one of the boxes must contain at least 2 elements.

Second Pigeonhole Principle

- If you put n objects into k boxes, $n > pk$, then one of the boxes must contain at least $p+1$ elements.

Permutations and Combinations

Permutations

Definition

- $S = \{a_1, \ldots, a_n\}$
- k-permutation of S = a sequence of k elements from S

  -- no duplicates (sampling without replacement)

  -- order DOES matter (ordered k-tuples)

- $P(n,k) = n(n-1)\cdots(n-k+1) = \dfrac{n!}{(n-k)!}$

  -- $k! = k(k-1)\cdots 1$

  -- $0! = 1$

Examples

- Number of fraternities with 3 *distinct* greek letters

  -- $P(24,3) = 24 * 23 * 22 = 12{,}144$

- Number of integers between 100 and 1000 with distinct odd digits

  -- $P(5,3) = 5 * 4 * 3 = 60$

More Examples

Linear Permutations

- Number of ways to seat n people at a long linear table

  -- $P(n,n) = n!$

Circular Permutations

- Number of ways to seat n people around a circular table

  -- $\dfrac{P(n,n)}{n} = (n-1)!$

  -- n circular shifts cause no relative change

  -- seat one person arbitrarily, then seat the others

Combinations

Definition

- $S = \{a_1, \ldots, a_n\}$
- k-combination of S = a subset of k elements from S
  - -- no duplicates (sampling without replacement)
  - -- order does NOT matter (subsets of k elements)
- $$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Card Games

- Number of 5 card poker hands
  - -- $$C(52,5) = \frac{52!}{5!\,47!} = \frac{52*51*50*49*48}{5!} = 2{,}598{,}960$$
- Number of 13 card bridge hands
  - -- $$C(52,13) = \frac{52!}{13!\,39!} > C(52,5)$$

More Examples

Probability of Splits in Bridge

- 4 Missing Trumps

    -- 2-2 split = $\dfrac{C(4,2)\,C(22,11)}{C(26,13)} \approx 40\%$

    -- 3-1 split = $\dfrac{2C(4,3)\,C(22,10)}{C(26,13)} \approx 50\%$

    -- 4-0 split = $\dfrac{2C(4,4)\,C(22,9)}{C(26,13)} \approx 10\%$

Taxicab Paths

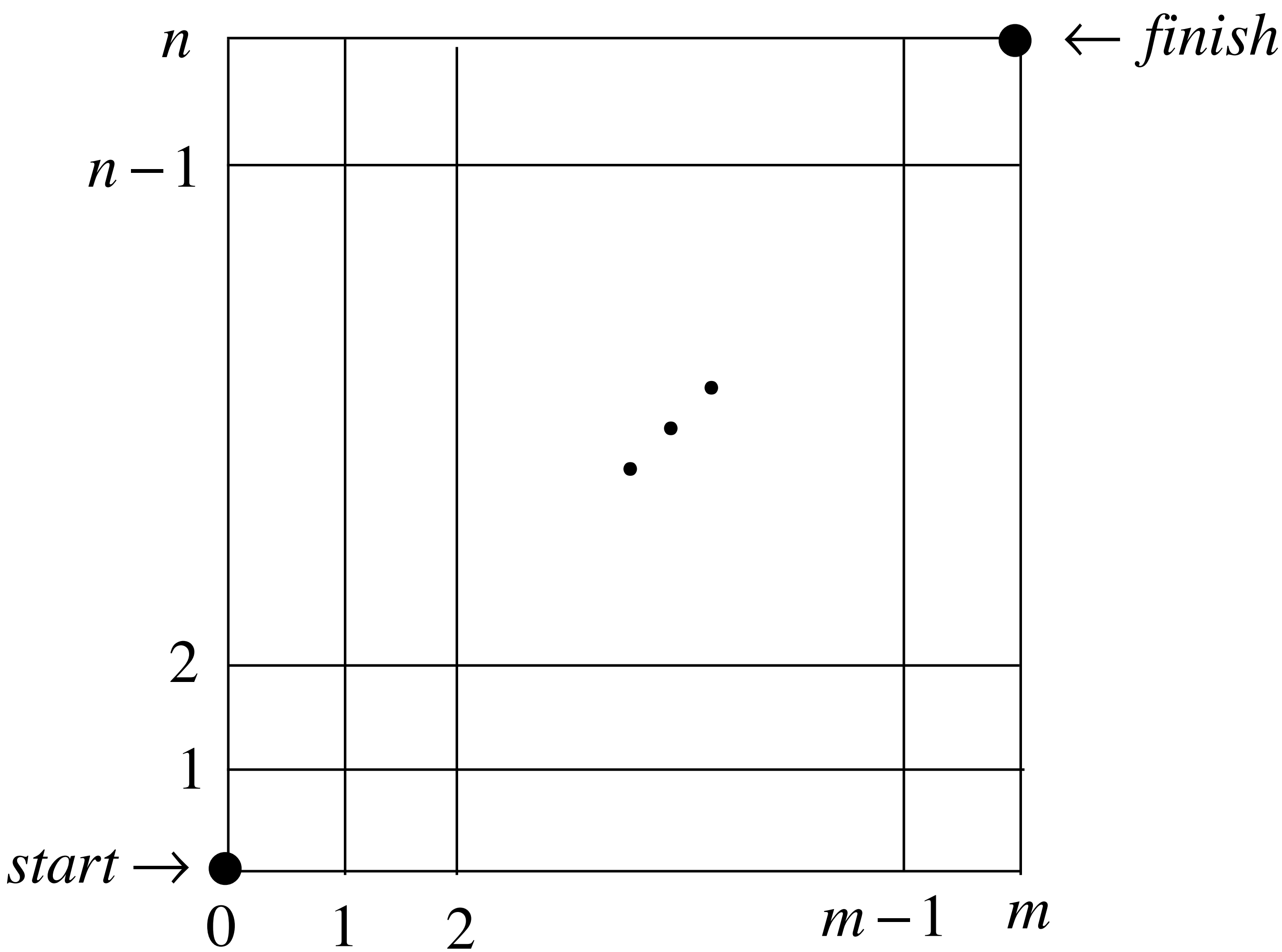
- Number of taxicab paths from $(0,0)$ to (m,n) along an integer grid m blocks east and n blocks north

    -- # paths = $C(m+n,n)$

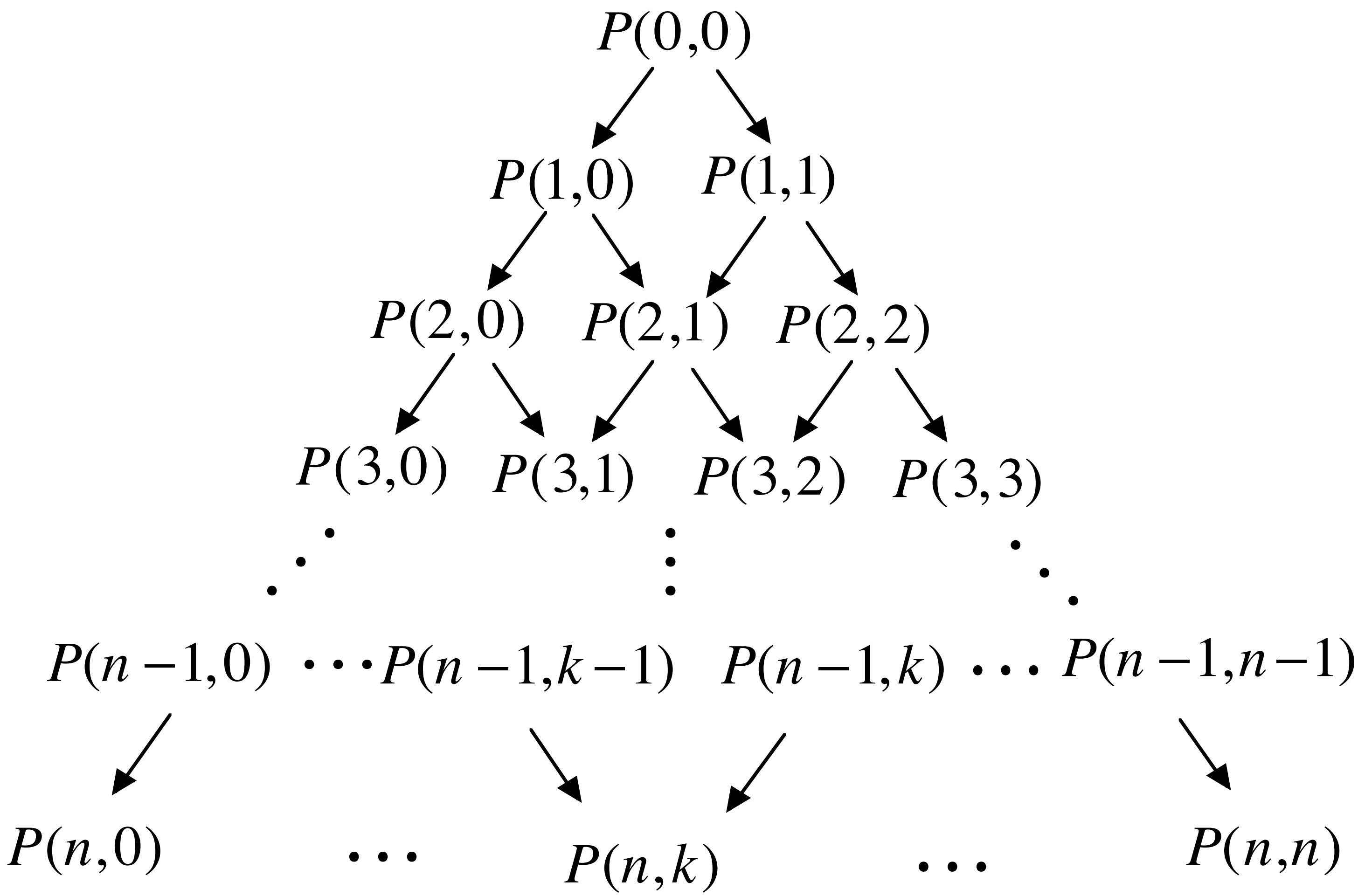
    -- must traverse m horizontal (east) and n vertical (north)

    -- must choose m horizontal paths from path of total length $m+n$

Paths in a Rectangle



Paths in a Triangle



Pascal's Triangle

								1								
							1		1							
						1		2		1						
					1		3		3		1					
				1		4		6		4		1				
			1		5		10		10		5		1			
		1		6		15		20		15		6		1		
	1		7		21		35		35		21		7		1	
1		8		28		56		70		56		28		8		1

$\vdots$

More Examples

Handshakes

- At a party of n people everyone shakes hands.

  -- # handshakes = $C(n,2) = \frac{n(n-1)}{2}$

Points in the Plane

- Consider n points in a plane, no two collinear

  -- # lines joining two points = $C(n,2) = \frac{n(n-1)}{2}$

  -- # triangles joining 3 points = $C(n,3) = \frac{n(n-1)(n-2)}{3!}$

Linear Equations with Unit Coefficients

Problem

- How many ways can you buy n computers of k different types, where you must buy at least one computer of each type?

Solution

- Let x_j = number of computers you buy of type j
- Find number of solutions in the *positive* integers of
  $x_1 + \cdots + x_k = n$

  -- List n 1's separated by $k-1$ spaces to form k sets: $x_1, \ldots, x_k$

  -- From $n-1$ potential spaces, choose $k-1$ spaces
- $C(n-1, k-1)$ = number of solutions in the *positive* integers

Indistinguishable Objects in Distinguishable Boxes

Problem

- How many ways can you put n similar objects into k different boxes at least one in each box?

Solution

- Let x_j = number of object you put into box j
- Find number of solutions in the *positive* integers of $x_1 + \cdots + x_k = n$
- $C(n-1, k-1)$ = number of solutions in the *positive* integers

Linear Equations with Constraints

Problem

- Find number of solutions in the *positive* integers of

  $$x_1 + \cdots + x_k = n$$

  Subject to the Constraints:

  $$x_1 > r_1, \ldots, x_k > r_k$$

Solution

- Let $y_1 = x_1 - r_1, \ldots, y_k = x_k - r_k$
- Solve $y_1 + \cdots + y_k = x_1 - r_1 + \cdots + x_k - r_k = n - r_1 - \cdots - r_k$
- # solutions = $C(n - r_1 - \cdots - r_k - 1,\ k - 1)$

Indistinguishable Objects in Distinguishable Boxes

Problem

- How many ways can you put n similar objects into k different boxes placing at least $r_j + 1$ object into box j?

Solution

- Start by placing r_j object into box j for each j.
- Find number of solutions in the *positive* integers of

  $y_1 + \cdots + y_k = n - r_1 - \cdots - r_k$
- # solutions = $C(n - r_1 - \cdots - r_k - 1,\ k - 1)$

Binomial Theorem

Positive Integer Exponents

- $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} C(n,k) x^k y^{n-k}$

- old proof -- induction on n

- new proof -- combinatorial proof

  -- coeff $x^k\, y^{n-k}$ = choose k factors of x from n factors of $(x+y)\cdots(x+y)$

Negative Integer Exponents

- $(y-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k y^{-(k+n)}$

- proof by generating functions -- later

Identities

Symmetry

- $C(n,k) = C(n,n-k)$ $\left\{\dfrac{n!}{k!(n-k)!}\right\}$
- Proof
  - -- Binomial Theorem -- symmetry in x and y
  - -- Subsets -- 1-1 correspondence between subsets with k elements and subsets with $n-k$ elements

Pascal's Identity -- Recursion

- $C(n, k) = C(n-1, k-1) + C(n-1, k)$
- Proof -- either you choose the first element or you don't

Pascal's Triangle

								1								
							1		1							
						1		2		1						
					1		3		3		1					
				1		4		6		4		1				
			1		5		10		10		5		1			
		1		6		15		20		15		6		1		
	1		7		21		35		35		21		7		1	
1		8		28		56		70		56		28		8		1

$\vdots$

Pascal's Triangle

Sum of Binomial Coefficients

- $$\sum_{k=0}^{n} C(n,k) = 2^n$$

  -- number of subsets of set with *n* elements is 2^n

  -- from the binomial theorem: $2^n = (1+1)^n = \sum_{k=0}^{n} C(n,k)\,(1)^k\,(1)^{n-k}$

Alternating Sum of Binomial Coefficients

- $$\sum_{k=0}^{n} (-1)^k C(n,k) = 0$$

  -- from the binomial theorem: $0 = (-1+1)^n = \sum_{k=0}^{n} C(n,k)(-1)^k(1)^{n-k}$

  -- obvious for *n* odd since: $(-1)^k C(n,k) + (-1)^{n-k} C(n,n-k) = 0$

Pascal's Triangle and Fibonacci Numbers

1

1 ↗

1 1

1 ↗

1 2 1

2 ↗

1 3 3 1

3 ↗

1 4 6 4 1

5 ↗

1 5 10 10 5 1

8 ↗

Pascal's Triangle and Fibonacci Numbers

Definitions

$$a_n = C(n,0) + C(n-1,1) + C(n-2,2) + \cdots + C(n/2, n/2)$$

$$a_{n-1} = \qquad C(n-1,0) + C(n-2,1) + \cdots + C(n/2, n/2-1) + C\big((n-1)/2, (n-1)/2\big)$$

$$\text{Sum} = C(n+1,0) + C(n,1) + C(n-1,2) + \cdots + C(n/2+1, n/2) + C\big((n+1)/2, (n+1)/2\big)$$

Conclusion

$$a_{n-1} + a_n = a_{n+1}$$

Vandermonde's Identity

Identity

- $C(n+m,k)=\sum_{i=0}^{k}C(n,i)\,C(m,k-i) \qquad k\le\min(m,n)$

Proof

- Choose i from the first n and $k-i$ from the last m

- see Concrete Mathematics p169 for interesting consequences

Identities -- Summary

- $C(n,k) = \frac{n!}{k!(n-k)!} = \binom{n}{k}$ *Definition*
- $C(n,k) = C(n,n-k)$ *Symmetry*
- $C(n,k) = C(n-1,k-1) + C(n-1,k)$ *Recursion*
- $\sum_{k=0}^{n} C(n,k) = 2^n$ *Sum*
- $\sum_{k=0}^{n} (-1)^k C(n,k) = 0$ *Alternating Sum*
- $C(n+m,k) = \sum_{i=1}^{k} C(n,i)\ C(m,k-i)$ *Vandermonde Identity*

Tricks of the Trade

1. Introduce Separators

2. Reason from a Model

3. Apply Binomial Theorem

Permutations and Combinations

with

Duplications and Indistinguishable Objects

Permutations with Duplication

Problem

- How many ways can we select n objects from k types
  - -- duplicates allowed (sampling with replacement)
  - -- order DOES matter
  - -- $n > k$ allowed

Solution

- $P = k^n$ -- follows from the product rule.

Examples

- 3 English Initials = 26^3
- Fraternities with 3 Greek Letters = $24^3 = 13{,}824$

Combinations with Duplication

Problem

- Choose n objects from k types
  - -- duplicates allowed (sampling with replacement)
  - -- order does NOT matter
  - -- $n > k$ allowed

Solution

- Imagine placing breaks || between the groups of objects *
  - -- n objects *
  - -- $k-1$ breaks || (breaks separate types)
  - -- total slots = objects + breaks = $n+k-1$
- From $n+k-1$ slots
  - -- choose exactly n slots for the objects or exactly $k-1$ slots for the breaks
  - -- $C(n+k-1, k-1) = C(n+k-1, n)$

Examples

Banana Splits

- 31 flavors of iced cream, 3 choices, repetitions allowed
- # banana splits = $C(33,3)$
- # banana splits with no duplicate flavors = ?   (2 ways)

Fruit Bowls

- 4 pieces of fruit from bowl with apples, oranges, pears
- # choices = $C(6,4)$

Nested Loops

Nested Loops

- For $j_1 = 1 \; to \; n$

  For $j_2 = 1 \; to \; j_1$

  $\ddots$

  For $j_m = 1 \; to \; j_{m-1}$

  Execute Some Actions

- Total Number of Times Through Loops = # Choices of $1 \le j_m \le j_{m-1} \le \cdots \le j_1 \le n$

- Choice of m objects -- $j_m, \ldots, j_1$ -- from n types -- numbers $1, \ldots, n$

- Total Number of Times Through Loops = $C(m+n-1,\, m) = C(m+n-1, n-1)$

Linear Equations with Unit Coefficients

Problem

- Find number of solutions in the *non-negative* integers of

  $x_1 + \cdots + x_k = n$

- Constraints: $x_1 \geq 0, \ldots, x_k \geq 0$

Solution

- # solutions = # ways of choosing n objects from k types

  $= C(n+k-1, k-1) = C(n+k-1, n)$

- # solutions = # ways of choosing $k+n$ objects from k types,

  choosing <u>at least one</u> of each type

  $= C(n+k-1, k-1) = C(n+k-1, n)$

Indistinguishable Objects in Distinguishable Boxes

Problem

- How many ways can you put n similar objects into k different boxes?

Solution

- Let x_j = number of object you put into box j
- Find number of solutions in the *nonnegative* integers of

  $x_1 + \cdots + x_k = n$
- $C(n+k-1, n) = C(n+k-1, k-1)$

Linear Equations with Constraints

Problem

- Find number of solutions in the *non-negative* integers of

  $x_1 + \cdots + x_k = n$

- Constraints: $x_1 \geq r_1, \ldots, x_k \geq r_k$

Solution

- Change of Variables:

  -- $y_1 = x_1 - r_1, \ldots, y_k = x_k - r_k$

- Solve

  -- $y_1 + \cdots + y_k = x_1 - r_1 + \cdots + x_k - r_k = n - r_1 - \cdots - r_k$

- # Solutions = $C(n + k - r_1 - \cdots r_k - 1,\ k - 1)$

  $$= C(n + k - r_1 - \cdots - r_k - 1, n - r_1 - \cdots - r_k)$$

Indistinguishable Objects in Distinguishable Boxes

Problem

- How many ways can you put n similar objects into k different boxes placing at least r_j object into box j?

Solution

- Start by placing r_j objects into box j for each j.
- Find number of solutions in the *nonnegative* integers of $y_1 + \cdots + y_k = n - r_1 - \cdots - r_k$
- $C(n + k - r_1 - \cdots r_k - 1,\ k - 1) = C(n + k - r_1 - \cdots - r_k,\ n - r_1 - \cdots - r_k)$

Special Case

- $r_j = 1$
- $C(n + k - r_1 - \cdots r_k - 1,\ k - 1) = C(n - 1,\ k - 1)$

Permutations with Indistinguishable Objects

Problem

- Find the number of permutations/combinations of n objects consisting of k classes of indistinguishable objects with $q_1, \ldots, q_k$ members. $\left(\sum_{j=1}^{k} q_j = n \right)$

  -- no duplicates (sampling without replacement)

  -- order matters only between the classes

Solution

- $P(n; q_1,\ldots,q_k) = \dfrac{n!}{q_1!\cdots q_k!} = C(n : q_1,\ldots,q_k)$

  -- $P(n; q_1,\ldots,q_k) = C(n, q_1)\, C(n - q_1, q_2)\cdots C(n - q_1 - \cdots q_{k-1}, q_k)$

  -- $q_1!\cdots q_k!\, P(n; q_1,\ldots,q_k) = n!$

Examples

Permutations of MISSISSIPPI

- $C(11;4,4,2,1) = \dfrac{11!}{4!4!2!1!}$

Placing n distinguishable objects into k boxes, q_j into box j.

- $C(n;q_1,\ldots,q_k) = C(n,q_1)\,C(n-q_1,q_2)\cdots C(n-q_1-\cdots q_{k-1},q_k)$
- $C(n;q_1,\ldots,q_k) = \dfrac{n!}{q_1!\cdots q_k!}$

Observation

- If $\sum q_j < n$, then $C(n;q_1,\ldots,q_k) = \dfrac{n!}{q_1!\cdots q_k!(n-q_1-\cdots-q_k)!}$

Multinomial Theorem

Multinomial Expansion

- $$(x_1 + \cdots + x_k)^n = \sum_{q_1 + \cdots + q_k = n} C(n;\, q_1, \ldots, q_k)\, x_1^{q_1} \cdots x_k^{q_k}$$

Proofs

- old proof -- induction on n
- new combinatorial proof

  -- coeff $(x_1)^{q_1} \cdots (x_k)^{q_k}$ = choose a permutation of $(x_1)^{q_1} \cdots (x_k)^{q_k}$ (consisting of q_j factors of x_j) from n factors of $(x_1 + \cdots + x_k)$

  -- $C(n;\, q_1, \ldots, q_k) = C(n, q_1)\, C(n - q_1, q_2) \cdots C(n - q_1 - \cdots q_{k-1}, q_k)$

Example

- $(x + y + z)^3 = x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3y^2z + 3yz^2 + 3x^2z + 3xz^2 + 6xyz$

More Examples

Problems

- How many seven letter words can you make from *barbara*?

  -- $C(7;2,2,3)=\frac{7!}{2!\,2!\,3!}=210$

- How many three letter words you can make from *anagram*?

  -- 0 *a*'s: $4!=24$

  -- 1 *a*: $3!\,C(4,2)=36$

  -- 2 *a's*: $3C(4,1)=12$

  -- 3 *a's*: 1

  -- Total 73

More Examples

Problems

- Number of ways to deal 4 bridge hands

  -- $C(52, 13, 13, 13, 13) = \dfrac{52!}{13!\,13!\,13!\,13!}$

Summary

1. $C(n,k)$ = # ways to choose k objects from n objects

2. $C(n+k-1, k-1)$ = # ways to choose n objects from k *inexhaustible* types

3. $C(n-1, k-1)$ = # ways to choose n objects from k *inexhaustible* types, selecting at least one of each type

4. $C(n; q_1, \ldots, q_k)$ = # ways to choose n objects from k types, where there are *exactly* q_j objects of type j.

Inclusion-Exclusion

Sum Rules

1. $|A \cup B| = |A| + |B|$ $\quad\quad A \cap B = \phi$

2. $|A \cup B| = |A| + |B| - |A \cap B|$

3. $$\begin{aligned} |A \cup (B \cup C)| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\ &= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \end{aligned}$$

4. $$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n)| &= \sum_{k=1}^{n} |A_k| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots \\ &\quad + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n| \end{aligned}$$

Inclusion-Exclusion

Rule

- $$|A_1 \cup A_2 \cup \cdots \cup A_n)| = \sum_{k=1}^{n} |A_k| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots$$
  $$+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Proof

- Let x an element be in r sets. On the right hand, x is counted

  $$C(r,1) - C(r,2) + \cdots + (-1)^{r+1} C(r,r) = C(r,0) = 1$$

Intuition

- Look at Venn Diagram for $n = 2, 3$

Sieve of Erastosthenes

Number of Primes Between 1 and 100

- $A = \{k \mid 1 \le k \le 100 \ \ and \ \ 2 \mid k\}$
- $B = \{k \mid 1 \le k \le 100 \ \ and \ \ 3 \mid k\}$
- $C = \{k \mid 1 \le k \le 100 \ \ and \ \ 5 \mid k\}$
- $D = \{k \mid 1 \le k \le 100 \ \ and \ \ 7 \mid k\}$

Primes between 1 and 100 = 99 – $|A \cup B \cup C \cup D|$ + 4

	$\|A\| = 50$	$\|A \cap B\| = 16$	$\|A \cap B \cap C\| = 3$	$\|A \cap B \cap C \cap D\| = 0$
	$\|B\| = 33$	$\|A \cap C\| = 10$	$\|A \cap B \cap D\| = 2$	
	$\|C\| = 20$	$\|A \cap D\| = 7$	$\|A \cap C \cap D\| = 1$	
	$\|D\| = 14$	$\|B \cap C\| = 6$	$\|B \cap C \cap D\| = 0$	
		$\|B \cap D\| = 4$		
		$\|C \cap D\| = 2$		
Totals:	117	45	6	0

Sieve of Erastosthenes (continued)

$$| A \cup B \cup C \cup D | = 117 - 45 + 6 - 0 = 78$$

Primes between 1 and 100 = 99 – 78 + 4 = 25

Linear Equations with Unit Coefficients

Problem

- Find number of solutions of $x_1 + \cdots + x_k = n$
- Constraints: $1 \le x_1 \le r_1, \ldots, 1 \le x_k \le r_k$

Recall

- $C(n-1, k-1)$ = number of solutions in the *positive* integers

Solution

- A_1 = solutions with $x_1 > r_1$ . . . A_k = solutions with $x_k > r_k$
- # Solutions = $C(n-1, k-1) - | A_1 \cup \cdots \cup A_k |$

Example

Equation and Constraints

- $x_1 + x_2 + x_3 + x_4 = 26$
- $1 \le x_1, x_2, x_3, x_4 \le 9$

Solution

- Total Solutions without Constraint = $C(n-1, k-1) = C(25, 3)$
- A_i = Solutions with $x_i \ge 10$

  -- $|A_i| = C(16, 3)$ $\qquad$ $y_i = x_i - 9 \ge 1$

  -- $|A_i \cap A_j| = C(7, 3)$

  -- $|A_i \cap A_j \cap A_k| = 0$

- #Constrained Solutions = $C(25, 3) - 4C(16, 3) + C(4, 2)C(7, 3)$

Observation

- Simpler solution later using Generating Functions

Onto Functions

Problem

- Find number of *onto* functions
  - -- $f: A \to B$,
  - -- $|A| = m$, $|B| = n$, and $m \ge n$

Recall

- # functions = n^m

Solution

- A_k = functions not having b_k in the range
  - -- $|A_k| = (n-1)^m$
  - -- $|A_i \cap A_j| = (n-2)^m$
  - -- $|A_{i1} \cap \cdots \cap A_{ik}| = (n-k)^m$
- # onto functions = $n^m - n(n-1)^m + C(n,2)(n-2)^m - \cdots \pm C(n,n-1)$
- # onto functions = $n^m + \sum_{k=1}^{n-1} (-1)^k C(n,k)(n-k)^m$

Derangements

Problem

- Derangement = Permutation where no elements appears in natural position
- Find $D(n) =$ number of Derangements of n elements

Example

- 2341, 4123, 3142, …

Solution

- A_k = solutions with k in natural position $k = 1, 2, \ldots, n$.
  - -- $|A_k| = (n-1)!$
  - -- $|A_i \cap A_j| = (n-2)!$
  - -- $|A_{i1} \cap \cdots \cap A_{ik}| = (n-k)!$
- $D(n) = n! - n(n-1)! + C(n,2)(n-2)! - \cdots \pm C(n,0)$
- $D(n) = \sum_{k=0}^{n} (-1)^k C(n,k)(n-k)! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
- $D(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots \pm \frac{1}{n!}\right) \approx \frac{n!}{e}$

Summary

Summary of Combinatorics

Counting Principles

Pigeonhole Principles

Permutations and Combinatorics

Tricks of the Trades

Placing Objects into Distinguishable Boxes

Combinations

Counting Principles -- Basic Rules

Sum Rules -- OR

- $|A \cup B| = |A| + |B| \qquad A \cap B = \phi$
- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A_1 \cup A_2 \cup \cdots \cup A_n)| = \sum_{k=1}^{n} |A_k| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$

Product Rules -- AND

- $|A \times B| = |A|\,|B|$
- $|A_1 \times A_2 \times \cdots \times A_n| = |A_1|\,|A_2| \cdots |A_n|$

Complement Rule -- NOT

- $A \subset S \Rightarrow |A| = |S| - |A^c|$

Pigeonhole Principles

First Pigeonhole Principle

- If you put n objects into k boxes, $n > k$, then one of the boxes must contain at least 2 elements.

Second Pigeonhole Principle

- If you put n objects into k boxes, $n > pk$, then one of the boxes must contain at least $p+1$ elements.

Permutations and Combinatorics -- Basic Formulas

- $P(n,k) = \dfrac{n!}{(n-k)!}$

- $C(n,k) = \dfrac{n!}{k!(n-k)!}$

- $P(n;q_1,\ldots,q_k) = \dfrac{n!}{q_1!\cdots q_k!}$

Tricks of the Trade

1. Introduce Separators
2. Reason from a Model
3. Use the Binomial Theorem
4. Apply Generating Functions

Placing Objects into Distinguishable Boxes

# *Objects*	*Boxes/Types*	# *Ways*
n *indistinguishable*	k $x_j \geq 1$ $x_1 + \cdots + x_k = n$	$C(n-1, k-1)$
n *indistinguishable*	k $x_j \geq 0$ $x_1 + \cdots + x_k = n$	$C(n+k-1, k-1)$
n *distinguishable*	k $q_1 + \cdots + q_k = n$	$C(n; q_1, \ldots, q_k)$

Placing Distinguishable Objects into Distinguishable Boxes

# Objects	*Boxes/Types*	*# Ways*
m	n	
distinguishable	*distinguishable*	$\sum_{k=0}^{n-1} (-1)^k C(n,k)(n-k)^m$
	$x_j \geq 1$	
	(Onto Functions)	

Combinations

- $C(n,k)$ = # ways to choose k objects from n objects

- $C(k+n-1,k-1)$ = # ways to choose n objects from k *inexhaustible* types

- $C(n-1,k-1)$ = # ways to choose n objects from k *inexhaustible* types,

  selecting at least one of each type

- $C(n;q_1,\ldots,q_k)$ = # ways to arrange n objects from k types,

  where here are *exactly* q_j objects of type j.

  $$= C(n,q_1)C(n-q_1,q_2)\cdots C(n-q_1-\cdots-q_{k-1},q_k)$$