Structural Induction

Examples

1. Propositions (Later)
   a. Base Case: T, F, p,q,r,...
   b. Recursive Step: ~p, p^q, p∨q, p→q

2. Polynomials
   a. Base Case: 1, x
   b. Recursive Step: $p+q, p*q, c\,p$

3. Binary Trees
   a. Base Case: Empty Tree, Tree with one node
   b. Recursive Step: Node with left and right subtrees

4. Strings (of Balanced Parentheses)
   a. Base Case: Empty string, ()
   b. Recursive Step: (S), S_1S_2

Principle of Structural Induction

Let R be a recursive definition.

Let S be a statement about the elements defined by R.

If the following hypotheses hold:

i. S is True for every element $b_1,\ldots,b_m$ in the base case of the definition R.

ii. For every element E constructed by the recursive definition from some elements $e_1,\ldots,e_n$:

   S is True for $e_1,\ldots,e_n \Rightarrow S$ is true for E

Then we can conclude that:

iii. S is True for Every Element E defined by the recursive definition R.

Template for Proofs by Structural Induction

Prove

i. S is True for $b_1,\ldots,b_n$ {Base Case}

ii. S is True for $e_1,\ldots,e_n \Rightarrow S$ is True for E {Inductive Step}

Conclude

iii. S is True for Every Element defined by R {Conclusion)

Observations on Structural Induction

Proofs by Structural Induction

- Extends inductive proofs to discrete data structures -- lists, trees,. . .
- For every recursive definition there is a corresponding structural induction rule.
- The base case and the recursive step mirror the recursive definition.
  - -- Prove Base Case
  - -- Prove Recursive Step

Proof of Structural Induction

Let $T = \{E \mid S \text{ is True for } E\}$.

- T contains the base cases
- T contains all structures that can be built from the base cases

Hence T must contain the entire recursively defined set.

Binary Trees

1. Recursive Definition
   a. Base Case: Empty Tree ϕ
   b. Recursive Step: Node with left and right subtrees

2. Structural Induction
   a. If $P(\phi)$ and $\forall T_1, T_2 \left\{ P(T_1) \wedge P(T_2) \Rightarrow P(T) \text{ with nodes } T_1, T_2 \right\}$
   b. Then $\forall T \, P(T)$

3. Size of a Tree
   a. Base Case: $s(\phi) = 0$
   b. Recursive Step: $s(T) = 1 + s(T_1) + s(T_2)$

4. Height of a Tree
   a. Base Case: $h(\phi) = 0$
   b. Recursive Step: $h(T) = 1 + \max\left(h(T_1),\, h(T_2)\right)$

Theorem: $s(T) \le 2^{h(T)+1} - 1$

Proof: By Structural Induction.

Base Case: $s(\phi) = 0$ and $h(\phi) = 0$

$$s(\phi) = 0 < 1 = 2 - 1 = 2^{h(\phi)+1} - 1$$

Recursive Step: Let T be the tree with nodes T_1, T_2

Assume: $s(T_1) \le 2^{h(T_1)+1} - 1$ and $s(T_2) \le 2^{h(T_2)+1} - 1$

Must Show: $s(T) \le 2^{h(T)+1} - 1$

Structural Induction: By definition

$$h(T) = 1 + \max\left(h(T_1),\ h(T_2)\right) = \max\left(1 + h(T_1),\ 1 + h(T_2)\right)$$

$$s(T) = 1 + s(T_1) + s(T_2)$$

Induction continued

$$s(T) = 1 + s(T_1) + s(T_2)$$

$$\leq 1 + \left(2^{h(T_1)+1} - 1\right) + \left(2^{h(T_2)+1} - 1\right) \qquad \text{\{Inductive Hypothesis\}}$$

$$\leq 1 + 2\left(2^{\max\left(h(T_1)+1,\ h(T_2)+1\right)} - 1\right)$$

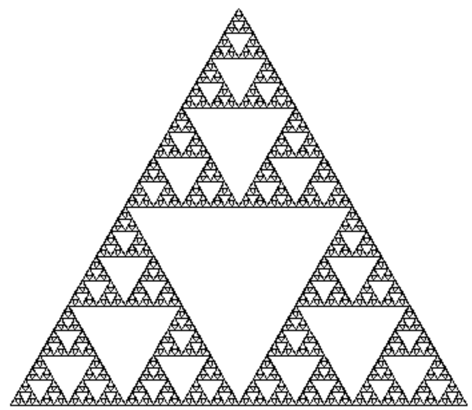
$$\leq 2\ \left(2^{h(T)}\right) - 1$$

$$= 2^{h(T)+1} - 1$$

Fractals

Theorem

Every angle in a Sierpinski Triangle is 60 degrees.



Proof

Base Case: Easy.

Inductive Step: By Structural Induction.

Balanced Parentheses

1. Definition
   a. Base Case: λ (empty string)
   b. Recursive Step: (S), S_1S_2

2. Structural Induction
   a. $P(\lambda)$ and $\forall S_1, S_2 \left\{ P[S_1]\ and\ P[S_2] \right\} \rightarrow P[(S_1)]\ and\ P[S_1S_2]$

      then $\forall S\, P[S]$

3. Count Function
   a. $c[S]$ = #open parentheses – #closed parentheses
      i. $c(\lambda) = 0$
      ii. $c[(S)] = c[S]$
      iii. $c[S_1S_2] = c[S_1] + c[S_2]$

Theorem: $c[S] = 0$

Proof: By Structural Induction.

Base Case: $c[\lambda] = 0$

Recursive Step;

$$c[(S)] = c[S] = 0$$

$$c[S_1 S_2] = c[S_1] + c[S_2] = 0 + 0 = 0$$

More Strings

Recursive Definition

- Base Cases: b
- Recursive Step: aSa

Explicit Formula

- $a^n b a^n \quad n \geq 0$

Structural Induction

- If $P(b)$ and $\left(\forall S \; P(S) \rightarrow P(aSa)\right)$, then $\forall S \, P(S)$

Theorem: Recursive Definition $\Leftrightarrow$ *Explicit Definition*

Proof: Recursive $\Rightarrow$ Explicit.

Every element constructed recursively is of the form $a^n b a^n$

By Structural Induction.

Base Case: $b = a^0 b a^0$.

Structural Induction:

- Suppose $S = a^n b a^n$
- Then $aSa = a(a^n b a^n)a = a^{n+1} b a^{n+1}$

Explicit $\Rightarrow$ Recursive.

Every element of the form $a^n b a^n$ can be constructed recursively.

By Weak Induction on n.

Base Case: $n = 0 \Rightarrow a^0 b a^0 = b$ Okay.

Induction

Assume: Every element of the form $a^n b a^n$ can be constructed recursively.

Must Show: Every element of the form $a^{n+1} b a^{n+1}$ can be constructed recursively.

Observe: $a^{n+1} b a^{n+1} = a(a^n b a^n)a = aSa$.

By the inductive hypothesis: $a^n b a^n$ satisfies the recursive definition;

Hence by the recursive step, so does $a^{n+1} b a^{n+1}$.

Polynomials

Recursive Definition

- Base Cases: $1, x$
- Recursive Step: $p+q,\ p*q,\ c\,p$

Explicit Definition

- $p(x) = c_0 + c_1 x + \cdots + c_n x^n$

Structural Induction

- If $S(1)$, $S(x)$ and $\left(\forall p,q\ S(p) \wedge S(q) \rightarrow S(p+q), S(p*q), S(c\,p)\right)$,

  then $\forall p\, S(p)$

Theorem: Recursive Definition $\Leftrightarrow$ *Explicit Definition*

Proof: Recursive $\Rightarrow$ Explicit.

Every element constructed recursively is of the form

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n .$$

By Structural Induction.

Base Case: 1, x . Okay.

Structural Induction:

- Suppose

  $$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

  $$q(x) = d_0 + d_1 x + \cdots + d_m x^m$$

- Then $p+q, p*q, c\,p$ are also of the form

  $$r(x) = e_0 + e_1 x + \cdots + e_k x^k$$

Explicit $\Rightarrow$ Recursive.

Every polynomial

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

can be constructed recursively.

By Weak Induction on n.

Base Case: $degree(p) = 0 \Rightarrow p = c1$. Okay.

Recursive Step: Suppose every polynomial

$$q(x) = c_0 + c_1 x + \cdots + c_n x^n$$

of degree n can be constructed recursively.

Must Show: Every polynomial

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n + c_{n+1} x^{n+1} = q(x) + c_{n+1} x^{n+1}$$

of degree $n+1$ can be constructed recursively.

By the inductive hypothesis: $q(x)$ and x^n are can both be constructed recursively

Hence by the recursive definition so can

$$c_{n+1} x^{n+1} = c_{n+1}(x x^n) \quad \text{and} \quad p(x) = q(x) + c_{n+1} x^{n+1}.$$

Structural Induction on the Natural Numbers

Recursive Definition

- Base Case: 0 is in $\boldsymbol{N}$
- Recursive Step: if n is in N, the $s(n) = n + 1$ is in $\boldsymbol{N}$

Observation

- Structural Induction $\Leftrightarrow$ Weak Induction

Theorem: Structural Induction on Recursive Schemes $\Leftrightarrow$ Weak Induction

Proof: $\Rightarrow$: Weak Induction follows from Structural Induction because weak induction is structural induction on the natural numbers.

$\Leftarrow$: Structural Induction follows from weak induction by induction on the number of operations = number of recursive steps.

<u>**Structural Induction on Pairs of Natural Numbers**</u>

Lexicographic Order on $N \times N$

- Think order on two letter words

  -- at, in, it, an

  -- $(2,3), (9,7), (2,7), (7,7)$

Well Ordering on $N \times N$

- Every non-empty subset of $N \times N$ has a smallest element.
- But there are infinitely many elements smaller than any element in $N \times N$

  -- List all elements less than $(4,7)$

Well Ordering on $N \times N$

Theorem: Every non-empty subset of $N \times N$ *has a smallest element.*

Proof: Let

$$S = \text{a nonempty subset of } N \times N.$$

$$S_1 = \{ s \in N \mid there\ is\ a\ number\ t\ such\ that\ (s,t) \in S \}$$

$$s^* = smallest\ element\ in\ S_1$$

$$S_2 = \left\{ t \in N \mid (s^*, t) \in S_1 \right\}$$

$$t^* = smallest\ element\ in\ S_2$$

Claim: $(s^*, t^*) = smallest\ element\ in\ S$

Proof: s^* *= smallest* s*, and for this smallest* s*,* t^* *= the smallest* t*.*

Strong Induction on Pairs of Natural Numbers

Let $P(m,n)$ be a statement about the pair of integers (m,n).

If the following hypotheses hold

i. Base Case: $P(0,0)$

ii. Recursive Step: $P(a,b)\ for\ all\ (a,b)<(c,d) \Rightarrow P(c,d)$

Then we can conclude that

iii. $P(m,n)$ is True for every pair of integers (m,n)

Proof: By Well Ordering Principle:

There is no smallest element where $P(m,n)$ is False.

Example

Recursive Definition

- $a_{0,0} = 0$
- $a_{m,0} = a_{m-1,0} + 1$
- $a_{m,n} = a_{m,n-1} + n \qquad n > 0$

Theorem: $a_{m,n} = m + n(n+1)/2$

Proof: By Strong Induction on $N \times N$.

Base Case: Obvious. $(0 = 0)$

Recursion: Two cases:

Case 1: $a_{m,0} = a_{m-1,0} + 1 = (m-1) + 1 = m$.

Case 2: $a_{m,n} = a_{m,n-1} + n = m + (n-1)n/2 + n = m + n(n+1)/2$.

Bad Recursive Definitions

Legal Definitions

- New objects must be built from objects already in the set

Incorrect Example: Strings with more 0's than 1's

- Base Case: 0
- Recursive Step: $0S, S0$, where S has same number of 0's and 1's.

Observation

- This recursive definition is not legal, since S is not in the set!
- Need to tell how S is constructed!