Strong Induction

<u>Principle of Strong Induction</u>

Let $P(n)$ be a statement about the *nth* integer.

If the following hypotheses hold:

i. $P(1)$ is True.

ii. The statement $P(1) \wedge P(2) \wedge \cdots \wedge P(n) \rightarrow P(n+1)$ is True for all $n \geq 1$.

Then we can conclude that:

iii. $P(n)$ is True for Every Integer $n \geq 1$.

<u>Template for Proofs by Strong Induction</u>

Prove

i. $P(1)$ {Base Case}

ii. $P(1) \wedge P(2) \wedge \cdots \wedge P(n) \to P(n+1)$ {Inductive Step}

Conclude

iii. $P(n)$ is True for Every Integer $n \geq 1$ {Conclusion)

Remark

- Do not always need to start at 1.

Examples -- Strong Induction

1. Game of NIM

2. Sums of Powers

3. Factoring Integers and Polynomials (and Complex Integers)

4. Egyptian Fractions

NIM

Game

- *P* Piles of Coins
- 2 Players
- Players Alternate Taking Coins from One Pile
- Player Forced to Take Last Coin Loses

Two Piles

- Equal Number of Coins in Each Pile
- Unequal Number of Coins in Each Pile

Variations

- Arbitrary Number of Piles
- Restrictions on Number of Coins Removed at Each Stage

<u>Sums of Powers</u>

0. $$\sum_{k=1}^{n} k^0 = n$$

1. $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

2. $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

3. $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n}{4}$$

Sums of Powers (continued)

Observations

1. $\sum_{k=1}^{n} k^p$ is a polynomial in n

2. $\sum_{k=1}^{n} k^p$ is divisible by n and $n+1$ if $p \geq 1$

3. $\sum_{k=1}^{n} k^p = \dfrac{n^{p+1}}{p+1}$ + lower order terms

<u>Sums of Powers (continued)</u>

Observations (continued)

4. $\displaystyle\sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} +$ lower order terms.

5. The sum of the coefficients of the powers of n is 1.

6. $\displaystyle\sum_{k=1}^{n} k^p$ factors into linear factors.

7. The roots of $\displaystyle\sum_{k=1}^{n} k^p$ are non-positive rational numbers.

Binomial Theorem

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Pascal's Triangle

								1								
							1		1							
						1		2		1						
					1		3		3		1					
				1		4		6		4		1				
			1		5		10		10		5		1			
		1		6		15		20		15		6		1		
	1		7		21		35		35		21		7		1	
1		8		28		56		70		56		28		8		1

⋮

Pascal -- 1654

Chu Shih-Chieh -- 1303

Binomial Theorem

Factorial

$$0! = 1$$

$$n! = n(n-1)! = n(n-1)\cdots 1$$

Binomial Coefficients

$$\binom{p}{r} = \frac{p!}{r!(p-r)!} \qquad 0 \le r \le p$$

$$= 0 \qquad r < 0 \quad or \quad r > p$$

Recurrence

$$\binom{p+1}{r} = \binom{p}{r} + \binom{p}{r-1}$$

Binomial Theorem

$$(a+b)^p = a^p + pa^{p-1}b + \cdots + \binom{p}{r}a^{p-r}b^r + \cdots + b^p = \sum_{r=0}^{p}\binom{p}{r}a^{p-r}b^r$$

<u>Binomial Coefficients</u>

Recurrence

$$\binom{p+1}{r} = \binom{p}{r} + \binom{p}{r-1}$$

Proof

$$\binom{p}{r} + \binom{p}{r-1} = \frac{p!}{r!\,(p-r)!} + \frac{p!}{(r-1)!\big(p-(r-1)\big)!}$$

$$= \frac{p!}{(r-1)!\,(p-r)!}\left\{\frac{1}{r} + \frac{1}{p+1-r}\right\}$$

$$= \frac{p!}{(r-1)!\,(p-r)!}\left\{\frac{(p+1-r)+r}{r\,(p+1-r)}\right\}$$

$$= \frac{(p+1)!}{r!(p+1-r)!} = \binom{p+1}{r}$$

Binomial Theorem

Theorem

$$(a+b)^p = a^p + pa^{p-1}b + \cdots + \binom{p}{r}a^{p-r}b^r + \cdots + b^p = \sum_{r=0}^{p}\binom{p}{r}a^{p-r}b^r$$

Proof

By weak induction on n.

Base Case: $(a+b)^1 = a+b = \binom{1}{0}a + \binom{1}{1}b$

Inductive Step: Assume $(a+b)^p = \sum_{r=0}^{p}\binom{p}{r}a^{p-r}b^r$

Must show: $(a+b)^{p+1} = \sum_{r=0}^{p+1}\binom{p+1}{r}a^{p+1-r}b^r$

Inductive Step (continued):

$$(a+b)^{p+1} = (a+b)(a+b)^p$$

$$= (a+b)\sum_{r=0}^{p}\binom{p}{r}a^{p-r}b^r \qquad \{Inductive\ Hypothesis\}$$

$$= a\sum_{r=0}^{p}\binom{p}{r}a^{p-r}b^r + b\sum_{r=0}^{p}\binom{p}{r}a^{p-r}b^r$$

$$= \sum_{r=0}^{p}\binom{p}{r}a^{p+1-r}b^r + \sum_{r=0}^{p}\binom{p}{r}a^{p-r}b^{r+1}$$

$$= \sum_{r=0}^{p}\binom{p}{r}a^{p+1-r}b^r + \sum_{s=1}^{p+1}\binom{p}{s-1}a^{p-(s-1)}b^s \qquad \{Reindexing\ Trick: s = r+1\}$$

$$= \sum_{r=0}^{p+1}\left\{\binom{p}{r}+\binom{p}{r-1}\right\}a^{p+1-r}b^r \qquad \{Reindex\ Again:\ s = r\}$$

$$= \sum_{r=0}^{p+1}\left\{\binom{p+1}{r}\right\}a^{p+1-r}b^r \qquad \left\{\mathrm{Re}\,currence\ \ for\ \binom{p+1}{r}\right\}$$

Binomial Theorem

Theorem

$$(a+b)^p = a^p + pa^{p-1}b + \cdots + \binom{p}{r}a^{p-r}b^r + \cdots + b^p = \sum_{r=0}^{p}\binom{p}{r}a^{p-r}b^r$$

Coefficients

$$\binom{p}{r} = \frac{p!}{r!(p-r)!} \qquad 0 \le r \le p$$

$$= 0 \qquad r < 0 \quad or \quad r > p$$

$$\binom{p}{r} = \binom{p-1}{r} + \binom{p-1}{r-1}$$

Binomial Theorem

Theorem

$$(a+b)^p = a^p + pa^{p-1}b + \cdots + \binom{p}{r}a^{p-r}b^r + \cdots + b^p = \sum_{r=0}^{p}\binom{p}{r}a^{p-r}b^r$$

Corollary

$$(a+1)^p = a^p + pa^{p-1} + \cdots + \binom{p}{r}a^{p-r} + \cdots + 1 = \sum_{r=0}^{p}\binom{p}{r}a^{p-r}$$

Sums of Powers

$$\sum_{k=0}^{n}(k+1)^{p+1}=\sum_{k=0}^{n}\left(k^{p+1}+\binom{p+1}{1}k^{p}+\cdots+\binom{p+1}{r}k^{p+1-r}+\cdots+k^{0}\right)$$

$$\sum_{k=0}^{n}(k+1)^{p+1}=\sum_{k=0}^{n}k^{p+1}+\binom{p+1}{1}\sum_{k=0}^{n}k^{p}+\cdots+\binom{p+1}{r}\sum_{k=0}^{n}k^{p+1-r}+\cdots+\sum_{k=0}^{n}k^{0}$$

$$\sum_{k=0}^{n}(k+1)^{p+1}-\sum_{k=0}^{n}k^{p+1}=\binom{p+1}{1}\sum_{k=0}^{n}k^{p}+\cdots+\binom{p+1}{r}\sum_{k=0}^{n}k^{p+1-r}+\cdots+\sum_{k=0}^{n}1$$

$$(n+1)^{p+1}=\binom{p+1}{1}\sum_{k=0}^{n}k^{p}+\cdots+\binom{p+1}{r}\sum_{k=0}^{n}k^{p+1-r}+\cdots+\sum_{k=0}^{n}1$$

$$(n+1)^{p+1}-\binom{p+1}{2}\sum_{k=0}^{n}k^{p-1}-\cdots-\binom{p+1}{r}\sum_{k=0}^{n}k^{p+1-r}-\cdots-\sum_{k=0}^{n}1=(p+1)\sum_{k=0}^{n}k^{p}$$

Sums of Powers

Notation

$$S_p(n) = \sum_{k=1}^{n} k^p$$

Observation

$$(p+1)\sum_{k=0}^{n} k^p = (n+1)^{p+1} - \binom{p+1}{2}\sum_{k=0}^{n} k^{p-1} - \cdots - \binom{p+1}{r}\sum_{k=0}^{n} k^{p+1-r} - \cdots - \sum_{k=0}^{n} 1$$

Conclusion

$$S_p(n) = \frac{(n+1)^{p+1} - \binom{p+1}{2}S_{p-1}(n) - \cdots - \binom{p+1}{r}S_{p+1-r}(n) - \cdots - \binom{p+1}{p}S_1(n) - (n+1)}{p+1}$$

<u>**Sums of Powers**</u>

Example: $p = 2$

$$S_2(n) = \frac{(n+1)^3 - 3S_1(n) - (n+1)}{3}$$

$$\sum_{k=1}^{n} k^2 = \frac{(n+1)^3 - 3n(n+1)/2 - (n+1)}{3} = (n+1)\left\{\frac{2(n+1)^2 - 3n - 2}{6}\right\}$$

$$\sum_{k=1}^{n} k^2 = (n+1)\left\{\frac{2n^2 + 4n + 2 - 3n - 2}{6}\right\} = \frac{n(n+1)(2n+1)}{6}$$

Properties of Sums of Powers

Properties

1. $S_p(n) = \sum_{k=1}^{n} k^p$ is a polynomial in n

2. $S_p(n) = \sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1}$ + lower order terms

3. $S_p(n) = \sum_{k=1}^{n} k^p$ is divisible by n and $n+1$ if $p \geq 1$

Proof

These results all follow by Strong Induction on p because

$$S_p(n) = \frac{(n+1)^{p+1} - \binom{p+1}{2} S_{p-1}(n) - \cdots - \binom{p+1}{r} S_{p+1-r}(n) - \cdots - \binom{p+1}{p} S_1(n) - (n+1)}{p+1}$$

<u>Sums of Powers (continued)</u>

Observations (continued)

4. $\sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} +$ lower order terms.

5. The sum of the coefficients of the powers of n is 1.

6. $\sum_{k=1}^{n} k^p$ factors into linear factors.

7. The roots of $\sum_{k=1}^{n} k^p$ are non-positive rational numbers.

Calculus

Differentiation

$$F'(x) = Lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

Integration

$$\int_0^1 F(x)\,dx = Lim_{n \to \infty} \sum_{k=1}^{n} F\left(\frac{k}{n}\right)\frac{1}{n}$$

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \{F'(x) = f(x)\}$$

Calculus for Polynomials

Differentiation

$$\frac{d}{dx}(x^{p+1}) = (p+1)x^p$$ (Induction from Product Rule)

Integration

$$\int_0^1 x^p \, dx = \left.\frac{x^{p+1}}{p+1}\right|_0^1 = \frac{1}{p+1}$$ (Fundamental Theorem of Calculus)

Differentiation for Polynomials

Theorem: $\frac{d}{dx}(x^p) = p\,x^{p-1}$ {p is a positive integer}

Proof: By Weak Induction.

Base Case: $Lim_{h \to 0} \frac{x + h - x}{h} = Lim_{h \to 0} \frac{h}{h} = 1 = 1\,x^0$

Inductive Step: Assume $\frac{d}{dx}(x^p) = p\,x^{p-1}$

Must Show: $\frac{d}{dx}(x^{p+1}) = (p+1)x^p$

Inductive Step

By the Product Rule.

$$\frac{d(fg)}{dx} = f\frac{dg}{dx} + g\frac{df}{dx}$$

Therefore, since $x^{p+1} = xx^p$:

$$\begin{aligned}\frac{d}{dx}(x^{p+1}) &= x\frac{d(x^p)}{dx} + x^p\frac{dx}{dx} \\ &= x(px^{p-1}) + x^p(1) \qquad \{\textit{inductive hypothesis}\} \\ &= (p+1)x^p\end{aligned}$$

Definite Integrals from Limits

Theorem: $\int_0^1 x^p\,dx = \frac{1}{p+1}$

Proof: $\int_0^1 x^p\,dx = Lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^p \left(\frac{1}{n}\right)$ *{Definition of Definite Integral}*

$$= Lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{k=1}^{n} k^p$$

$$= Lim_{n\to\infty} \frac{1}{n^{p+1}} \left\{ \frac{n^{p+1}}{p+1} + lower\ order\ terms \right\}$$

$$= \frac{1}{p+1}$$

Commentary

1. Weak induction proof typically goes up from n to $n+1$.

2. Strong induction (recursion) typically goes down from $n+1$ to lower values.

   a. Strong induction may seem more natural -- conceptually easier.

   b. Strong induction has a more constructive/programming feel.