

# PARTIIIc

Version 2.14: 2004/05/14 14:23:24.565 GMT-5

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## Abstract

Interpretations map relation-symbols to an actual relation.

## partIIIc

### 1 Needing Interpretations to Evaluate Formulas

You might have noticed something funny: we said  $\text{safe}(a)$  depended on the board, but that  $\text{prime}(18)$  was false. Why are some relations different than others? To add to the puzzling, there was a caveat in some fine-print from the previous section: " $\text{prime}(18)$  is false *under the standard interpretation of prime*". Why these weasel-words? Everybody knows what prime is, don't they? Well, if our domain is matrices of integers (instead of just integers), we might suddenly want a different idea "prime".

Consider the formula  $E(x, x)$  true for all  $x$  in a domain? Well, it depends not only on the domain, but also the specific binary relation  $E$  actually stands for:

- for the domain of integer where  $E$  is interpreted as "even", this is false;
- for the domain  $\{2, 4, 6, 8\}$  where  $E$  is interpreted as "even", this is true;
- for the domain of integers where  $E$  is interpreted as "greater than", this is false;
- for the domain of people where  $E$  is interpreted as "is at least as tall as", this is true.

Thus a formula's truth depends on the interpretation of the (syntactic, meaning-free) relation symbols in the formula.

#### Definition 1: Interpretation

The interpretation of a formula is a domain, and a mapping from the formula's relation symbols to specific relations on the domain.

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### 1.1 A non-standard interpretation (optional)

Note that there are other possible interpretations of "prime". For example, since one can multiply integer matrices, there might be a useful concept of "prime matrices".

For example: Consider *only* the numbers  $F = \{1, 5, 9, 13, \dots\}$  – that is,  $F = \{4k+1 \mid k \text{ in } \mathbb{N}\}$ . It's easy to verify that multiplying two of these numbers still results in a number of the form  $4k+1$ . Thus it makes sense to talk of factoring such numbers: We'd say that 45 factors into  $5*9$ , but 9 is considered prime since it doesn't factor into smaller elements of  $F$ .

Interestingly, within  $F$ , we lose *unique* factorization:  $441 = 9*49 = 21*21$ , where each of 9, 21, and 49 are prime, relative to  $F$ ! (Mathematicians will then go and look for exactly what property of a multiplication function are needed, to guarantee unique factorization.)

The point is, that all relations in logical formula need to be interpreted. Usually, for numbers, we use a standard interpretation, but one can consider those formulas in different, non-standard interpretations!

### 1.2 Using Formulas to Classify Interpretations (Optional)

In the previous section (pg ??), having a formula was rather useless until we had a particular interpretation for it. But we can view that same idea backwards: Given a formula, what are all the interpretations for which the formula is true?

For instance, consider a formula expressing that an array is sorted ascendingly: For all numbers  $i, j$ ,  $((i < j) \rightarrow \text{element}(i) \leq \text{element}(j))$ . But if we now broaden our mind about what relations/functions the symbols  $\text{element}$ ,  $<$ ,  $\leq$ , and then wonder about the set of all structures/interpretations which make this formula true, we might find that our notion of sorting is broader than we first thought. (Or equivalently, we might decide that the notion "ascending" applies to more structures than we first suspected).

Similarly, mathematicians create some formulas about functions being associated and having an identity elements, and then look for structures which have that property, saying this is what defines (say) "groups".

### 1.3 Encoding functions as Relations

What about adding functions, to our language, in addition to relations? Well, functions are just a way of relating input(s) to an output. For example, 3 and 9 are related by the square function, as are 9 and 81, and 0,0. Is any binary relation a function? No – for instance  $\{(9,81), (9,17)\}$  is not a function, because there is no *unique* output related to the input 9.

How can we enforce uniqueness? The following sentence asserts that for each element  $x$  of the domain,  $R$  associates at most one value with  $x$ : For all  $x, y$  and  $z$  of the domain,

$$((R(x, y) \wedge R(x, z)) \rightarrow y = z) \tag{1}$$

This is a common trick, for to describe uniqueness: if  $y$  and  $z$  each have some property, then they must be equal. (We have not yet specified that for every element of the domain, there is *at least* one element associated with it; we'll get to that later.)

#### Exercise 1:

We just used a binary relation to model a unary function. Carry on this idea, by using a ternary relation to start to model a binary function. In particular, write a formula stating that for every pair of elements  $w, x$  in the domain, the relation  $S$  associates at most one value with that pair.

#### Solution:

For all  $w$ ,  $x$ ,  $y$  and  $z$  of the domain,

$$((S(w, x, y) \wedge S(w, x, z)) \rightarrow y = z) \quad (2)$$