partIVb: Boolean algebra for first-order logic

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Abstract

Extend the laws of boolean algebra from propositional logic to first-order logic.

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Now that we can express interesting concepts using the quantifiers "$\exists$" ("there exists") and "$\forall$" ("for all"), how can we use them for the problem of determining whether a formula is true? Back in lowly propositional logic, we had three methods:

- truth tables,
- equivalences, and
- formal proofs with inference rules.

How can we adapt these approaches, for first-order logic?

Well, truth tables have no analog approach. With quantifiers, we don't have a finite set of propositions. Furthermore, variables can't refer to specific items in the domain until we try to interpret them. And when we do, the domain may be of any size – possibly even infinite. Using a truth table on an infinite universe is clearly infeasible, but the real problem stems from how we want to be able to discuss reasoning without respect to a particular domain.

However, we can add equivalences and inference rules to cope with quantifiers. After showing how to work with quantifiers, we'll come back to examine our newly-augmented systems for those desirable traits, soundness and completeness.

1 First-order Equivalences

When we upgrade from propositional logic to first-order logic, what changes do we need to make to the laws of boolean algebra? Well first off, we can keep all the existing propositional equivalences[1] (.ps)[2]. For example, $\forall x.\, \neg(\phi \wedge \psi) \equiv \forall x.\, (\neg\phi \vee \neg\psi)$. (Technically, we're even

[1]http://cnx.rice.edu/content/m10540/latest/
[2]http://www.teachLogic.org/Base/Printables/algebra-laws.ps

making those equivalences stronger, since those meta-variables ϕ, ψ, θ can now stand for any *first-order* formula, rather than merely propositional formulas.)

But, we need additional identities to deal with our new-fangled quantifiers. What should these be? The most interesting are those that relate the two kinds of quantifiers. Universal quantification ($\forall$) says that something holds for all members of the universe, and existential quantification ($\exists$) says that something holds for at least one member. Clearly, $\forall x.\phi$ implies $\exists x.\phi$, but the other direction doesn't hold, so that is not an equivalence.

> ASIDE: Wait just a minute! That implication holds only if the universe is non-empty, so that there is at least one member in it. We'll see this restriction appear a few times.

What about $\forall x.\ \neg\phi$? In English, "for all items x, $\phi(x)$ does not hold". A more natural way to say this is that there is *no* item x such that $\phi(x)$ *does* hold – that is, $\neg\exists x.\phi$. Indeed, this will be one of our new boolean algebra rules.

See a list of equivalences with quantifiers[3] (.pdf)[4] (.ps)[5]. As before, we can use these to show other pairs of formulas equivalent, as in the following examples.

Example 1:

Using these identities, we can simplify formulas such as the following: $(\forall y.\forall x.\,(R\,(x) \wedge Q\,(x,y)) \wedge \neg\exists z.\ \neg R\,(z)\ )$.

1	$(\forall y.\forall x.\,(R\,(x) \wedge Q\,(x,y)) \wedge \neg\exists z.\ \neg R\,(z)\ )$	
2	$\equiv (\forall y.\forall x.\,(R\,(x) \wedge Q\,(x,y)) \wedge \forall z.\ \neg\neg R\,(z)\ )$	Complementation of $\exists$
3	$\equiv (\forall y.\forall x.\,(R\,(x) \wedge Q\,(x,y)) \wedge \forall z.R\,(z))$	Double Complementation
4	$\equiv (\forall x.\forall y.\,(R\,(x) \wedge Q\,(x,y)) \wedge \forall z.R\,(z))$	Reordering $\forall$s
5	$\equiv (\forall x.\,(R\,(x) \wedge \forall y.Q\,(x,y)) \wedge \forall z.R\,(z))$	Distribution of $\forall$ over $\wedge$
6	$\equiv (\forall x.\,(R\,(x) \wedge \forall y.Q\,(x,y)) \wedge \forall x.R\,(x))$	renaming
7	$\equiv \forall x.\,((R\,(x) \wedge \forall y.Q\,(x,y)) \wedge R\,(x))$	Distribution of $\forall$ over $\wedge$
8	$\equiv \forall x.\,((\forall y.Q\,(x,y) \wedge R\,(x)) \wedge R\,(x))$	Commutativity of $\wedge$
9	$\equiv \forall x.\,(\forall y.Q\,(x,y) \wedge (R\,(x) \wedge R\,(x)))$	Associativity of $\wedge$
10	$\equiv \forall x.\,(\forall y.Q\,(x,y) \wedge R\,(x))$	Idempotency of $\wedge$

As a reminder, small obvious steps such as the use here of commutativity and associativity are often omitted. We show them here to emphasize that the identities of propositional logic are also used in first-order logic.

Exercise 1:

The equivalences for distributing implication over equivalences seem counterintuitive at first glance. Show that the following one holds, given all the identities which don't involve both implication and quantifiers.

Assuming that ψ does not have any free occurrences of variable x, $\forall x.\,(\phi \rightarrow \psi) \equiv (\exists x.\phi \rightarrow \psi)$.

Solution:

[3] http://cnx.rice.edu/content/m11045/latest/
[4] http://www.teachLogic.org/Base/Printables/first-order-equivalences.pdf
[5] http://www.teachLogic.org/Base/Printables/first-order-equivalences.ps

1	$\forall x.\,(\phi \rightarrow \psi)$	
2	$\equiv \forall x.\,(\neg\phi \vee \psi)$	Definition of $\rightarrow$
3	$\equiv (\forall x.\,\neg\phi \vee \psi)$	Distribution of $\forall$ over $\vee$
4	$\equiv (\neg\exists x.\phi \vee \psi)$	Complementation of $\exists$
5	$\equiv (\exists x.\phi \rightarrow \psi)$	Definition of $\rightarrow$

Are the following two sentences true?

- "All flying pigs wear top hats." $\forall x \in \text{FlyingPigs}, \mathit{wears\_top\_hat}\,(x)$
- "All numbers in the empty set are even." $\forall x \in \{\}, \mathit{even}\,(x)$

Each sentence states that some property holds for every member of some set (flying pigs or the empty set), but there are no such members. Such sentences are considered **vacuously true**.

But, why should they be considered true? Consider their negations:

- $\exists s \in \text{FlyingPigs}, \neg\mathit{wears\_top\_hatx}$, "There exists a flying pig not wearing a top hat." (The negation is *not* "No flying pigs wear top hats.")
- $\exists x \in \{\}, \neg\mathit{evenx}$, "There exists a number in the empty set that is even." (The negation is *not* "No numbers in the empty set are even.")

These negations are clearly false, so the original sentences must be true. Alternatively, consider a series of statements:

- If S is a set of three even numbers, each number in S is even.
- If S is a set of two even numbers, each number in S is even.
- If S is a set of one even number, each number in S is even.

So, it seems natural to say that if S is a set of zero even numbers, each number in S is even. (Of course, if S is empty, each number in S is also odd!) This is also similar to the fact that a simple propositional implication, $(a \rightarrow b)$ is true, if a is in fact false, regardless of the truth of b. In the above sentences, the non-existence of members in the set corresponds to the falsity of a.

2 Are we done yet?

While equivalences are very useful, we are often interested in implications such as the one mentioned previously: $(\forall x.\phi \rightarrow \exists x.\phi)$. We could rephrase that as an equivalence, $(\forall x.\phi \rightarrow \exists x.\phi) \equiv \text{true}$. Informally, it should be clear that that is rather awkward, and formally it is as well.

But such implications are exactly what inference rules are good for. So, let's continue and consider what first-order inference rules[6] should be.

[6] http://cnx.rice.edu/content/m10774/latest/