Problem 1

a) For noninteracting capacities with linear resistances subject to a unit-step change in the input of the first tank, the material balance can be written as follows:

$$A_1R_1\frac{dy_1}{dt}+y_1=R_1u(t) \tag{S1.1}$$

$$A_2R_2\frac{dy_2}{dt}+y_2=\frac{R_2}{R_1}y_1 \tag{S1.2}$$

subject to the initial conditions

$$y_1(0)=y_{1s}=R_1u_s \qquad y_2(0)=y_{2s}=R_2/R_1\,y_{1s}=R_2u_s \tag{S1.3}$$

Defining the deviation variables $Y_1=y_1-y_{1s}$, $Y_2=y_2-y_{2s}$ and $Q=u(t)-u_s$, equations (S1.1)-(S1.2) become:

$$A_1R_1\frac{dY_1}{dt}+Y_1=R_1Q \tag{S1.4}$$

$$A_2R_2\frac{dY_2}{dt}+Y_2=\frac{R_2}{R_1}Y_1 \tag{S1.5}$$

subject to the initial conditions

$$Y_1(0)=0 \qquad Y_2(0)=0 \tag{S1.6}$$

Hence, the transfer functions for equations (S1.4)-(S1.5) are:

$$G_1(s)=\frac{\bar{Y}_1(s)}{\bar{Q}(s)}=\frac{R_1}{(A_1R_1)s+1} \tag{S1.7}$$

$$G_2(s)=\frac{\bar{Y}_2(s)}{\bar{Y}_1(s)}=\frac{R_2/R_1}{(A_2R_2)s+1} \tag{S1.8}$$

Since the two noninteracting tanks are placed in series, the overall transfer function is the following:

$$G(s)=G_1(s)G_2(s)=\frac{\bar{Y}_2(s)}{\bar{Q}(s)}=\frac{R_2}{\left[(A_1R_1)s+1\right]\left[(A_2R_2)s+1\right]} \tag{S1.9}$$

Rewriting (S1.9) in the standard form, gives:

$$G(s)=G_1(s)G_2(s)=\frac{\bar{Y}_2(s)}{\bar{Q}(s)}=\frac{R_2}{\left[(A_1R_1)(A_2R_2)s^2+2(A_1R_1+A_2R_2)+1\right]} \tag{S1.10}$$

For critically damped systems, ξ=1. Therefore, comparing equation (S1.10) with the standard form yields:

$$\tau^2 = \tau_1\tau_2 = (A_1R_1)(A_2R_2) \tag{S1.11}$$

$$\tau = \tau_1 + \tau_2 = A_1R_1 + A_2R_2 \tag{S1.12}$$

Equations (S1.12)-(S1.11) are simultaneously solved if:

$$\tau_1 = \tau_2 \quad \Rightarrow \quad A_1R_1 = A_2R_2 \tag{S1.13}$$

In other words, the system is critically damped if the roots of the denominator in (S1.9) are in reality one root with multiplicity equal to 2. This immediately implies (S1.13). Thus:

$$\frac{R_1}{R_2} = \frac{A_2}{A_1} = \frac{1}{2} \tag{S1.14}$$

b) From the textbook, solution to the critically damped second order system subject to a unit-step change is:

$$\frac{Y_2(t)}{R_2} = 1 - e^{-\frac{t}{\tau}}\left(1 + \frac{t}{\tau}\right) \tag{S1.15}$$

Since it takes 1 min for the change in level of the second tank to reach 50 percent of the total change, one obtains:

$$0.5 = 1 - e^{-\frac{1}{\tau}}\left(1 + \frac{1}{\tau}\right) \quad \Rightarrow \quad \tau = 0.59\,\text{min} \tag{S1.16}$$

c) From the textbook, solution to a first order system (1st tank) subject to a unit-step change is:

$$\frac{Y_1(t)}{R_1} = 1 - e^{-\frac{t}{\tau}} \tag{S1.17}$$

Therefore, in order for the level of the first tank to reach 90 percent of the total change it takes:

$$0.9 = 1 - e^{-\frac{t}{0.59}} \quad \Rightarrow \quad t = 0.59\ln(10)\,\text{min} = 1.36\,\text{min} \tag{S1.18}$$

Problem 2

Bringining the transfer function in standard form, yields:

$$G(s) = \frac{20}{4s^2 + 0.6s + 1} \tag{S2.1}$$

Therefore the natural period of oscillation and the damping factor are the following:

$$\tau^2 = 4 \quad \Rightarrow \quad \tau = 2 \text{ (time)} \tag{S2.2}$$

$$2\tau\xi = 0.6 \text{ (time)} \quad \Rightarrow \quad \xi = 0.6/(2\tau) = 0.15 \text{ (D-less)} \tag{S2.3}$$

Hence, from the page 191 of textbook one obtains:

$$OS = \exp\left(\frac{-\pi\xi}{\sqrt{1-\xi^2}}\right) = 0.62 \text{ (D-less)} \tag{S2.4}$$

$$\nu = \frac{\omega}{2\pi} = \frac{\sqrt{1-\xi^2}}{2\pi\tau} = 0.079 \frac{\text{cicles}}{\text{time}} \tag{S2.5}$$

Problem 3

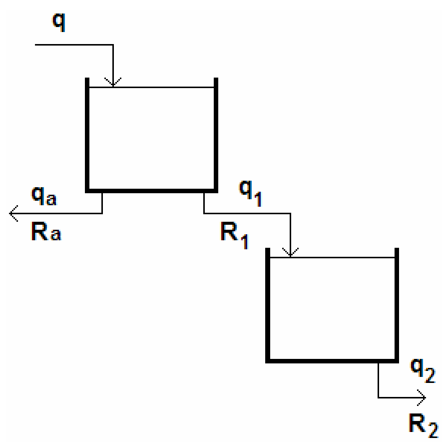

Figure 3.1

Let $1/R_t = 1/R_a + 1/R_1 = 3/2$. Then, the material balance on the first tank yields:

$$A_1 \frac{dh_1}{dt} = q - (q_A + q_1) = q - \frac{h_1}{R_t} \tag{S3.1}$$

subject to the initial condition $h_1(0) = h_{1s} = q_s R_t$. Equation (3.1) can be rewritten as follows:

$$A_1 R_t \frac{dh_1}{dt} + h_1 = R_t q \tag{S3.2}$$

The material balance on the second tank yields:

$$A_2 R_2 \frac{dh_2}{dt} + h_2 = \frac{R_2}{R_1} h_1 \tag{S3.3}$$

subject to the initial condition $h_2(0) = R_2/R_1 h_{1s}$. Defining the deviation variables $H_1 = h_1 - h_{1s}$, $H_2 = h_2 - h_{2s}$ and $Q = q - q_s$, equations (S3.2)-(S3.3) become:

$$A_1 R_t \frac{dH_1}{dt} + H_1 = R_t Q \tag{S3.4}$$

$$A_2 R_2 \frac{dH_2}{dt} + H_2 = \frac{R_2}{R_1} H_1 \tag{S3.5}$$

subject to the initial conditions

$$H_1(0)=0 \qquad\qquad H_2(0)=0 \tag{S3.6}$$

Hence, the transfer functions for equations (S3.4)-(S3.5) are:

$$G_1(s)=\frac{\bar{H}_1(s)}{\bar{Q}(s)}=\frac{R_t}{(A_1R_t)s+1}=\frac{2/3}{4/3s+1} \tag{S3.7}$$

$$G_2(s)=\frac{\bar{H}_2(s)}{\bar{H}_1(s)}=\frac{R_2/R_1}{(A_2R_2)s+1}=\frac{1}{s+1} \tag{S3.8}$$

Thus, the overall transfer function is the following:

$$G(s)=G_1(s)G_2(s)=\frac{\bar{H}_2(s)}{\bar{Q}(s)}=\frac{2/3}{\left(4/3s+1\right)\left(s+1\right)} \tag{S3.9}$$

Problem 4

a) Linearizing c_Ac_R about $c_{As}c_{Rs}$ yields:

$$\begin{aligned} C_AC_R &= C_{As}C_{Rs}+\left.\frac{\delta(C_AC_R)}{\delta C_A}\right|_{\substack{C_A=C_{As}\\ C_R=C_{Rs}}}(C_A-C_{As})+\left.\frac{\delta(C_AC_R)}{\delta C_R}\right|_{\substack{C_A=C_{As}\\ C_R=C_{Rs}}}(C_R-C_{Rs})+HOT= \\ &\simeq C_{As}C_{Rs}+C_{Rs}(C_A-C_{As})+C_{As}(C_R-C_{Rs}) \end{aligned} \tag{S4.1}$$

Substituting (S4.1) into (4.1a) and (4.2b) gives:

$$\tau\frac{dC_A}{dt}=C_{Ai}-C_A-kC_{As}C_{Rs}-kC_{Rs}(C_A-C_{As})-kC_{As}(C_R-C_{Rs}) \tag{S4.2}$$

$$\tau\frac{dC_R}{dt}=C_{Ri}-C_R+kC_{As}C_{Rs}+kC_{Rs}(C_A-C_{As})+kC_{As}(C_R-C_{Rs}) \tag{S4.3}$$

At steady state, one obtains:

$$C_{Ais}-C_{As}=kC_{As}C_{Rs} \tag{S4.4}$$

$$-C_{Ris}+C_{Rs}=kC_{As}C_{Rs} \tag{S4.5}$$

Defining the deviation variables $A=C_A-C_{As}$, $R=C_R-C_{Rs}$, $Q_A=C_{Ai}-C_{Ais}$ and $Q_R=C_{Ri}-C_{Ris}$ equations (S4.2)-(S4.3) become:

$$\tau\frac{dA}{dt}=Q_A-A-kC_{Rs}A-kC_{As}R \tag{S4.6}$$

$$\tau\frac{dR}{dt}=Q_R-R+kC_{Rs}A+kC_{As}R \tag{S4.7}$$

Subject to:

$$A(0) = 0 \quad R(0) = 0 \tag{S4.8}$$

b) Taking the Laplace transform of both sides of (S4.6)-(S4.7) and applying the initial conditions (S4.8) yields:

$$\tau s\bar{A}(s) = \bar{Q}_A(s) - \bar{A}(s) - kC_{Rs}\bar{A}(s) - kC_{As}\bar{R}(s) \tag{S4.9}$$

$$\tau s\bar{R}(s) = \bar{Q}_R(s) - \bar{R}(s) + kC_{Rs}\bar{A}(s) + kC_{As}\bar{R}(s) \tag{S4.10}$$

Notice that in equation (S4.9) $Q_R(s)=0$ since C_{Ri} is constant. Thus, solving (S4.10) for A gives:

$$\bar{A}(s) = \frac{\tau s + 1 - kC_{As}}{kC_{Rs}}\bar{R}(s) \tag{S4.11}$$

which substituted into (S4.9) provides the transfer function between R(s) and $Q_A(s)$:

$$G(s) = \frac{\bar{R}(s)}{\bar{Q}_A(s)} = \frac{kC_{Rs}}{s^2 + \frac{2}{\tau}\left[1 + \frac{k(C_{Rs} - C_{As})}{2}\right]s + \left[\frac{1 + k(C_{Rs} - C_{As})}{\tau^2}\right]} \tag{S4.12}$$

c) Notice that the denominator of G(s) can be written as:

$$as^2 + bs + c \tag{S4.13}$$

where a=1. The condition $(C_{As}-C_{Rs})<1/k$ ensures that the 1st order and 0th order coefficient of the polynomial denominator of G(s) are positive. Hence, since $s_1 s_2=c>0$, then the two roots have the same sign (either positive or negative). Moreover, as $-(s_1+s_2)=b$, then the two roots are both negative. Therefore, the condition $(C_{As}-C_{Rs})<1/k$ guarantees stability for the system.

d) From (S4.12) one obtains:

$$\xi = 1 + \frac{k(C_{Rs} - C_{As})}{2} \tag{S4.14}$$

Since the term $C_{Rs}-C_{As}$ is positive, so is the term $k(C_{Rs}-C_{As})/2$. Therefore, the damping factor ξ is bigger than 1 and the system in overdamped.