Problem 1

Overall Material Balance:

$$\frac{dn}{dt}=\frac{d(\rho V)}{dt}=\frac{d(\rho Ah)}{dt}=\rho_i F_i-\rho F \quad\Rightarrow\quad \rho A\frac{dh}{dt}+Ah\frac{d\rho}{dt}=\rho_i F_i-\rho F \qquad \text{(S1.1)}$$

In absence of chemical reactions and at constant temperature and pressure, the density of the fluid liquid in the tank can be assumed constant. Therefore, (S1.1) yields:

$$A\frac{dh}{dt}=F_i-F=F_i-20h^{1/2} \qquad \text{(S1.2)}$$

subject to the initial condition h(0)=h_S. At steady state, equation (S1.2) provides a functional relationship between the level asymptotically attained by the liquid in the tank and the stationary inlet flow rate:

$$\frac{dh}{dt}=0 \quad\Rightarrow\quad F_s=20h_s^{1/2} \quad\Rightarrow\quad h_s=9\text{ft} \qquad \text{(S1.3)}$$

The process model in (S1.3) is nonlinear. In order to linearize this model, one can expand the function F_i-$20h^{1/2}$ in (S1.3) about the steady state value F_s-$20h_S^{1/2}$ as follows:

$$\begin{aligned}F_i-20h^{1/2}&=F_s-20h_s^{1/2}+\left[\frac{\partial\left(F_i-20h^{1/2}\right)}{\partial F_i}\right]_{\substack{F_i=F_S\\h=h_S}}\left(F_i-F_s\right)+\left[\frac{\partial\left(F_i-20h^{1/2}\right)}{\partial h_i}\right]_{\substack{F_i=F_S\\h=h_S}}\left(h-h_s\right)+HOT=\\&\cong 0+[1]\left(F_i-F_s\right)-\left[10h_s^{-1/2}\right]\left(h-h_s\right)=\left(F_i-F_s\right)-\left[10h_s^{-1/2}\right]\left(h-h_s\right)\end{aligned} \qquad \text{(S1.4)}$$

Substituting equation (S1.4) into equation (S1.2) and defining the deviation variables H=h-h_S and R=F_i-F_S provides the linearized process model in deviation form:

$$A\frac{dH}{dt}=R-\frac{10}{3}H \qquad \text{(S1.5)}$$

subject to the initial condition H(0)=0. Let 10/3 be equal to B. Then, solution to equation (S1.5) can be obtained by employing the Laplace transform, as follows:

$$A\left[s\overline{H}(s)-H(0)\right]=\frac{R}{s}-B\overline{H}(s) \quad\Rightarrow\quad \overline{H}(s)=\frac{R}{s(As+B)}=\frac{Q(s)}{P(s)} \qquad \text{(S1.6)}$$

Notice that P(s) has two simple (multiplicity=1) roots s=0 and s=-B/A. By using partial fraction expansion, (S1.6) can be rewritten as follows:

$$\overline{H}(s)=\frac{R}{s(As+B)}=\frac{Q(s)}{P(s)}=\frac{C_1}{s}+\frac{C_2}{s+B/A} \qquad \text{(S1.7)}$$

where

$$\begin{aligned}C_1&=\left[s\overline{H}(s)\right]_{s=0}=\left[\frac{R}{As+B}\right]_{s=0}=R/B\\C_2&=\left[(As+B)\overline{H}(s)\right]_{s=-B/A}=\left[\frac{R}{s}\right]_{s=-B/A}=-AR/B\end{aligned} \qquad \text{(S1.8)}$$

Therefore, (S1.7) becomes:

$$\overline{H}(s) = \frac{R}{B}\left(\frac{1}{s} - \frac{A}{s + B/A}\right) \tag{S1.9}$$

Using table 7.1 from the book and taking the inverse Laplace transform of (S1.9) yields:

$$H(t) = h(t) - h_s = \frac{R}{B}\left[1 - A\exp(-B/A\,t)\right] \tag{S1.9}$$

or

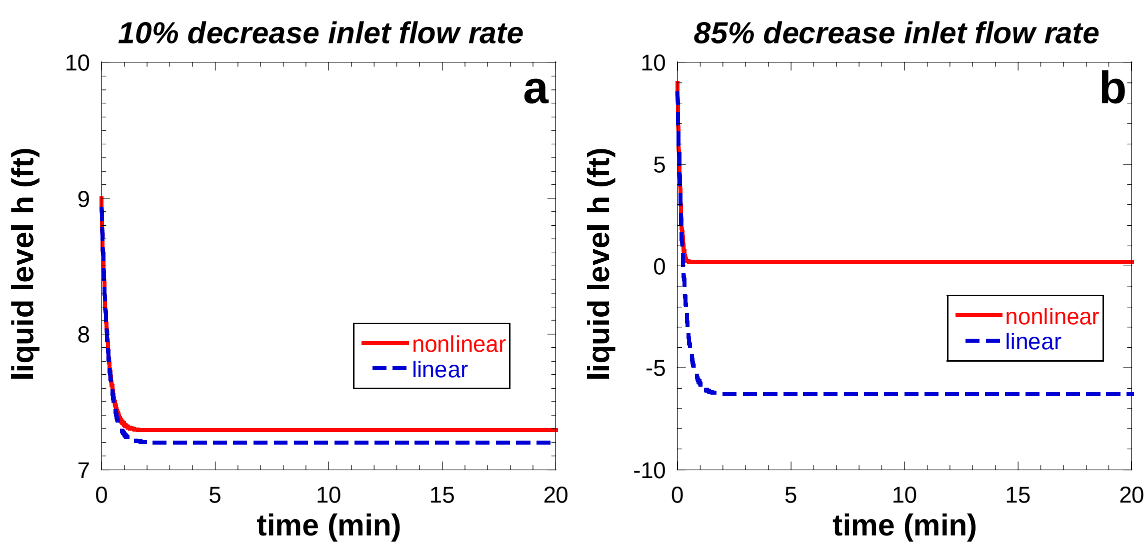
$$h(t) = 9 + \frac{3(F_i - 60)}{10}\left[1 - \exp\left(-\frac{10}{3}t\right)\right] \tag{S1.10}$$

For a 10% decrease in the inlet flow rate, $(F_i\text{-}60)=-0.1F_s=-6$ ft^3/min. Moreover, for an 85% decrease in the inlet flow rate, $(F_i\text{-}60)=-0.85F_s=-51$ ft^3/min. Therefore, the time-dependent change of the liquid level in the tank due to i) a 10% decrease and ii) an 85% decrease in the input flow rate is respectively given by:

10%: $$h(t) = 9 - 1.8\left[1 - \exp\left(-\frac{10}{3}t\right)\right] \tag{S1.11}$$

85% $$h(t) = 9 - 15.3\left[1 - \exp\left(-\frac{10}{3}t\right)\right] \tag{S1.12}$$

The comparison between the dynamics of the nonlinear model and those of the linearized model is illustrated in figure 1a and 1b for the case of 10% and 85% decrease in the inlet flow rate respectively.



Notice that for the case of 10% decrease, the discrepancy from the nonlinear model is small and the linear model can be considered adequate. However, for 85% decrease in the inlet flow rate, the discrepancy is huge. Thus, for large disturbances the linearization fails. Notice also that the new steady state predicted by the liner model is negative, which does not even make physical sense.

Problem 2

Invoking the linearity property of the Laplace transform, one can write:

$$L\left[(t-1)^2\right] = L\left[t^2 - 2t + 1\right] = L\left[t^2\right] - 2L[t] + L[1] \qquad \text{(S2.1)}$$

From table 7.1 in the book, we have:

$$L\left[t^n\right] = \frac{n!}{s^{n+1}}$$
$$L[a] = \frac{a}{s} \qquad \text{(S2.2)}$$

Therefore, for the particular case in exam, it follows that

$$L\left[t^2\right] = \frac{2}{s^3}$$
$$L[t] = \frac{1}{s^2} \qquad \text{(S2.3)}$$
$$L[1] = \frac{1}{s}$$

which yields:

$$L\left[(t-1)^2\right] = \frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s} = \frac{s^2 - 2s + 2}{s^3} \qquad \text{(S2.4)}$$

Problem 3

1. Take the Laplace transform of equation (3.1) and apply derivation theorem

$$\left[s^2\overline{x}(s) - sx(0) - \dot{x}(0)\right] + 5\left[s\overline{x}(s) - x(0)\right] + 6\overline{x}(s) = 1/s^2 \qquad \text{(S3.1)}$$

2. Apply initial conditions

$$(s^2 + 5s + 6)\overline{x}(s) = \frac{1}{s^2} \quad \Rightarrow \quad \overline{x}(s) = \frac{1}{s^2(s+2)(s+3)} \qquad \text{(S3.2)}$$

3. Identify roots and their multiplicity

$$s_1 = 0 \qquad \mu_1 = 2$$
$$s_2 = -2 \qquad \mu_2 = 1 \qquad \text{(S3.3)}$$
$$s_3 = -3 \qquad \mu_3 = 1$$

4. Partial fraction expansion

$$\overline{x}(s) = \frac{1}{s^2(s+2)(s+3)} = \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{C_3}{(s+2)} + \frac{C_4}{(s+3)} \qquad \text{(S3.4)}$$

5. Compute the coefficients

$$
\begin{aligned}
C_1 &= \left[\frac{1}{(s+2)(s+3)}\right]_{s=0} = \frac{1}{6} \\
C_2 &= \frac{d}{ds}\left[\frac{1}{(s+2)(s+3)}\right]_{s=0} = \left[\frac{-2s+5}{(s^2+5s+6)^2}\right]_{s=0} = -\frac{5}{36} \\
C_3 &= \left[\frac{1}{s^2(s+3)}\right]_{s=-2} = \frac{1}{4} \\
C_4 &= \left[\frac{1}{s^2(s+2)}\right]_{s=-3} = -\frac{1}{9}
\end{aligned}
\tag{S3.5}
$$

Therefore, (S3.4) can be rewritten as:

$$\bar{x}(s) = \frac{1/6}{s^2} - \frac{5/36}{s} + \frac{1/4}{s+2} - \frac{1/9}{s+3} \tag{S3.6}$$

Comment on the qualitative nature of the solution

The roots s=-2 and s=-3 are both on the real axis and on the left of the imaginary axis. Therefore, these roots are stabilizing as they will contribute to exponentially decaying terms. The root s=0 has multiplicity bigger than one and lies on the origin. This root is destabilizing as it contributes to a term t that increases in time. Therefore, the entire system will be ultimately unstable as t approaches infinite. This result can be double checked by applying the final value theorem and evaluating the limit for t going to infinite of x(t):

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0}\left[s\bar{x}(s)\right] = \lim_{s\to 0}\left[\frac{1/6}{s} - \frac{5}{36} + \frac{1/4\,s}{s+2} - \frac{1/9\,s}{s+3}\right] = \infty \tag{S3.7}$$

From table 7.1 in the book, we have:

$$L^{-1}\left[\frac{1}{(s+a)^{n+1}}\right] = \frac{t^n e^{-at}}{n!} \tag{S3.8}$$

Thus, inverting (S3.6) while considering (S3.8) gives:

$$x(t) = 1/6\,t - 5/36 + 1/4\,e^{-2t} - 1/9\,e^{-3t} \tag{S3.9}$$

Problem 4

1. Take the Laplace transform of equation (4.1) and apply derivation theorem

$$\left[s^4\bar{x}(s) - s^3x(0) - s^2\dot{x}(0) - s\ddot{x}(0) - \dddot{x}(0)\right] + \left[s^3\bar{x}(s) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0)\right] = s/(s^2+1) \tag{S4.1}$$

2. Apply initial conditions

$$s^3(s+1)\bar{x}(s) = \frac{s}{(s^2+1)} + (s+1) \quad \Rightarrow \quad \bar{x}(s) = \frac{s^3+s^2+2s+1}{s^3(s+1)(s^2+1)} \tag{S4.2}$$

3. Identify roots and their multiplicity

$$\begin{aligned} s_1 &= 0 & \mu_1 &= 3 \\ s_2 &= -1 & \mu_2 &= 1 \\ s_3 &= -j & \mu_3 &= 1 \\ s_4 &= +j & \mu_4 &= 1 \end{aligned} \tag{S4.3}$$

4. Partial fraction expansion

$$\bar{x}(s) = \frac{s^3+s^2+2s+1}{s^3(s+1)(s^2+1)} = \frac{C_1}{s^3} + \frac{C_2}{s^2} + \frac{C_3}{s} + \frac{C_4}{(s+1)} + \frac{C_5}{(s+j)} + \frac{C_6}{(s-j)} \tag{S4.4}$$

5. Compute the coefficients

$$\begin{aligned} C_1 &= \left[\frac{s^3+s^2+2s+1}{(s+1)(s^2+1)}\right]_{s=0} = 1 \\ C_2 &= \frac{d}{ds}\left[\frac{s^3+s^2+2s+1}{(s+1)(s^2+1)}\right]_{s=0} = 1 \\ C_3 &= \frac{1}{2}\frac{d^2}{ds^2}\left[\frac{s^3+s^2+2s+1}{(s+1)(s^2+1)}\right]_{s=0} = -1 \\ C_4 &= \left[\frac{s^3+s^2+2s+1}{s^3(s^2+1)}\right]_{s=-1} = \frac{1}{2} \\ C_5 &= \left[\frac{s^3+s^2+2s+1}{s^3(s+1)}\right]_{s=-j} = \frac{1}{4}(1-j) \\ C_6 &= \left[\frac{s^3+s^2+2s+1}{s^3(s+1)}\right]_{s=j} = \frac{1}{4}(1+j) \end{aligned} \tag{S4.5}$$

Therefore, (S4.4) can be rewritten as:

$$\bar{x}(s) = \frac{1}{s^3} + \frac{1}{s^2} - \frac{1}{s} + \frac{1/2}{(s+1)} + \frac{1/4(1-j)}{(s+j)} + \frac{1/4(1+j)}{(s-j)} \tag{S4.6}$$

Comment on the qualitative nature of the solution

The root s=0 has multiplicity 3 and will contribute to terms t^2 and t. Thus, this root contributes to divergence. The pair of complex and conjugates is simple and lies on the imaginary axis. Therefore, theses roots contribute to oscillations with constant amplitude. Finally, the root s=-1 is simple and lies on the left of the imaginary axis. Hence, this root contributes to an exponentially decaying term. The overall behavior of the system is unstable. Applying the final value theorem gives:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0}\left[s\bar{x}(s)\right] = \lim_{s\to 0}\left[\frac{1}{s^2}+\frac{1}{s}-1+\frac{1/2\,s}{(s+1)}+\frac{1/4(1-j)\,s}{(s+j)}+\frac{1/4(1+j)s}{(s-j)}\right] = \infty \qquad \text{(S4.7)}$$

Inverting (S4.6) while considering (S3.8) gives:

$$x(t) = 1/2\,t^2 + t - 1 + 1/2\,e^{-t} + 1/4\,(1-j)e^{-jt} + 1/4\,(1+j)e^{jt} \qquad \text{(S4.8)}$$

Applying the Euler identities $e^{jt} = \cos t + j\sin t$ and $e^{-jt} = \cos t - j\sin t$, (S4.8) assumes its final form:

$$x(t) = 1/2\,t^2 + t - 1 + 1/2\,e^{-t} + 1/2\left(\cos t - \sin t\right) \qquad \text{(S4.9)}$$