Chapter 6 Finite Difference Solution in Multidimensions

The partial differential equations for multiphase fluid flow derived in the previous section can be numerically solved by employing finite difference approximations for the partial differential equations. The finite difference procedure evaluates the dependent variables (pressure and saturation) at discrete points in space and in time. The derivatives are approximated by a difference of the dependent variable between two or more discrete points in space or in time.

The accuracy of the finite difference approximation is dependent on the formulation of the procedure and the size of the space or time increment. The dependence of the accuracy of the approximation on the size of the time or space increment can be estimated by a Taylor's series expansion. The effect of the size of the space or time increment on the reservoir performance can be directly determined by simulating the performance with different size increments.

The finite difference approximation should also be formulated such that it is consistent with the material balance over the entire reservoir over any discrete time interval. This will be discussed in a following section.

6.1 Approximation of Time Derivative

The solutions to the multiphase equations (pressure and saturations) are computed at discrete time intervals,

$$t = t_0, t_1, t_2, \ldots t_n, t_{n+1}, \ldots \tag{6.1a}$$

where t_0 is the initial time when the initial conditions are specified. The time increment from t_n to t_{n+1} is denoted as Δt.

$$\begin{aligned} t_{n+1} - t_n &= \Delta t_{n+1} \\ &= \Delta t \end{aligned} \tag{6.1b}$$

The value of the time increment, Δt_{n+1} can be arbitrarily specified each time step independent of the size of the previous time increment. With this being understood, the subscript n + 1 will be omitted from Δt_{n+1}.

A time derivative appears in the accumulation term of the material conservation equations.

$$\frac{\partial C}{\partial t} = -\nabla \cdot \vec{f} + \tilde{q} \tag{6.1c}$$

The generalized concentration, *C*, is of the form

$$C = \phi b S \tag{6.1d}$$

where the porosity and the reciprocal formation volume factor change with time as a function of pressure.

The partial derivative of *C* with time can be expressed as

$$\begin{aligned} \frac{\partial C}{\partial t} &= \frac{\partial \left(\phi b S\right)}{\partial t} \\ &= \phi b \frac{\partial S}{\partial t} + S \frac{d\left(\phi b\right)}{d\, p} \frac{\partial p}{\partial t} \end{aligned} \tag{6.1e}$$

The finite difference approximation of the partial derivative of *C* should be formulated such that it is consistent with the material balance. The finite difference approximation of the derivative can be approximated as

$$\frac{\partial C}{\partial t} \approx \frac{C_{n+1} - C_n}{\Delta t} \tag{6.1f}$$

By substituting the equation for *C* into the difference approximation, the difference approximation for *C* can be expressed in terms of the difference approximation of *S* and *p* .

$$\begin{aligned} \frac{C_{n+1} - C_n}{\Delta t} &= \frac{\phi_{n+1} b_{n+1} S_{n+1} - \phi_n b_n S_n}{\Delta t} \\ &\equiv \phi_{n+1} b_{n+1} \frac{\left(S_{n+1} - S_n\right)}{\Delta t} + S_n \frac{\left(\phi_{n+1} b_{n+1} - \phi_n b_n\right)}{\left(p_{n+1} - p_n\right)} \frac{\left(p_{n+1} - p_n\right)}{\Delta t} \end{aligned} \tag{6.1g}$$

This equation is not a unique formulation for the difference approximation to C, but it is a formulation that is consistent with the material balance and evaluates the coefficients of the time difference of *S* and *p* such that it is convenient to solve the system of equations.

The symbol, Δ_t will be used to denote the time change of a variable, e.g.,

$$\begin{aligned} \Delta_t \left(\phi b S\right) &\equiv \phi_{n+1} b_{n+1} S_{n+1} - \phi_n b_n S_n \\ \Delta_t S &\equiv S_{n+1} - S_n \end{aligned} \tag{6.1h}$$

6.2 Equal Grid Spacing

The finite difference approximation to the conservation equations relates the dependent variables (pressure and saturations) at a number of discrete grid points or grid block in the reservoir. A grid system in the x-y plane with equal grid spacing may appear as in Figure 6.1. The grid points are denoted by the solid circles in the center of each grid block, The grid points are connected by the dashed lines.

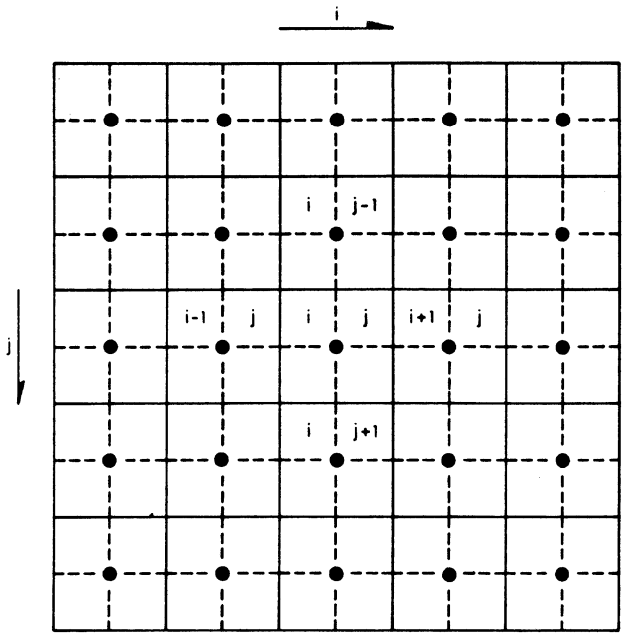

Fig. 6.1 Equal grid spacing

The spatial derivatives appear in the conservation equations in Cartesian coordinates as (assuming that the coordinate directions coincide with the principal directions of the permeability tensor.)

$$\nabla \cdot \left[\frac{\mathbf{k}\, k_r\, b}{\mu} \left(\nabla p - \rho\, g\, \nabla D \right) \right] =$$
$$+ \frac{\partial}{\partial x} \left[\frac{k_x\, k_r\, b}{\mu} \left(\frac{\partial p}{\partial x} - \rho\, g \frac{\partial D}{\partial x} \right) \right]$$
$$+ \frac{\partial}{\partial y} \left[\frac{k_y\, k_r\, b}{\mu} \left(\frac{\partial p}{\partial y} - \rho\, g \frac{\partial D}{\partial y} \right) \right]$$
$$+ \frac{\partial}{\partial z} \left[\frac{k_z\, k_r\, b}{\mu} \left(\frac{\partial p}{\partial z} - \rho\, g \frac{\partial D}{\partial z} \right) \right]$$

The derivative of pressure in the x -direction at x_i can be approximated as

$$\frac{\partial}{\partial x} \left(K_x \frac{\partial p}{\partial x} \right) \approx \frac{K_{x,i+1/2} \left(p_{i+1} - p_i \right) - K_{x,i-1/2} \left(p_i - p_{i-1} \right)}{\Delta x^2}$$

where

$$K_x = \frac{k_x\, k_r\, b}{\mu}$$

The finite difference equation for the spatial derivative of p at x_i has values of p at x_{i-1}, x_i, and x_{i+1} as shown on Figure 6.2.

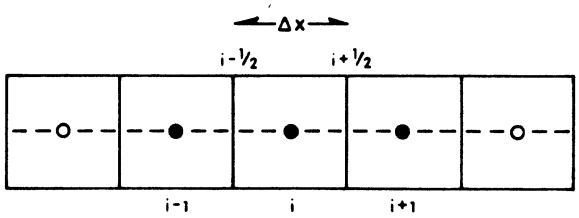

Fig. 6.2 Grid increments in the x-direction

The truncation error for the approximation will be evaluated for the case when $K = 1$.

$$\left(\frac{\partial^2 p}{\partial x^2}\right)_i = \frac{p_{i+1} - 2p_i + p_{i-1}}{\Delta x^2} - \frac{\Delta x^2}{24}\left[\frac{\partial^4 p}{\partial x^4}(\bar{x}) + \frac{\partial^4 p}{\partial x^4}(\bar{\bar{x}})\right] \tag{6.2c}$$

Thus, the truncation error of the finite difference approximation for the case of constant K and equal grid spacing is proportional to the square of Δx.

The terms $K_{i-1/2}$ and $K_{i+1/2}$ must be computed from the values at the grid points. The relative permeability is a function of the saturation and is computed from the upstream grid point or points. The pressure dependent quantities, b and μ, are just computed as an average between adjacent grid points. The permeability is computed as a harmonic mean value between the two grid blocks.

$$k_{i+1/2} = 2\frac{k_i\, k_{i+1}}{k_i + k_{i+1}} \tag{6.2d}$$

The harmonic average is the result from assuming uniform permeability in each grid block and one-dimensional flow. The result is equivalent to the average conductance of two resistances in series, The permeability, k_X, is the permeability in the x-direction.

The finite difference equation can be multiplied by the product, $\Delta x \Delta y \Delta z$, which is equal to the bulk volume of the grid block, to express it in the form

$$(\Delta x \Delta y \Delta z)\frac{\partial}{\partial x}\left(K_x \frac{\partial p}{\partial x}\right)_i \approx T_{x,i+1/2}(p_{i+1} - p_i) - T_{x,i-1/2}(p_i - p_{i-1})$$

where (6.2e)

$$T_{x,i+1/2} = \left(\frac{k_x\, k_r\, b}{\mu}\frac{\Delta y \Delta z}{\Delta x}\right)_{i+1/2}$$

The coefficient, T_X , has been denoted as an interblock transmissibility coefficient. It may be interpreted as a product of the following factors

$$T_x = \lambda_x b \frac{A_x}{L_x}$$

where

$$A_x = \Delta y \Delta z \qquad (6.2f)$$

$$L_x = \Delta x$$

$$\lambda_x = \frac{k_x k_r}{\mu}$$

It can be expressed in practical units as

$$T_x = \frac{1}{887.2} \frac{k_x k_r b}{\mu} \frac{\Delta y \Delta z}{\Delta x} \qquad (6.2g)$$

where

T_X = interblock transmissibility coefficient, (STB/D)/psi or (Mcf/D)/psi
k_X = permeability in x-direction, md
k_r = relative permeability
b = reciprocal formation volume factor, STB/RB or Mcf/RB
μ = viscosity, cp
$\Delta x, \Delta y, \Delta z$ = grid spacing, ft.

Thus, the product of the interblock transmissibility coefficient at $x_{i+l/2}$ and the pressure drop, $p_{i+1} - p_i$, represents the flow in STB/D or Mcf/D from the *i+1* grid block into the *i* grid block (see Figure 6.2) due to the pressure gradient.

The no-flow boundary conditions are imposed by using a zero value of the transmissibility at the boundaries. By using this method for specifying the boundary condition, the finite difference equations using the boundary condition is the same expression as for the interior grid blocks.

The notation for the differences will be condensed by using the Δ symbol to denote differences.

$$T_{x,i+1/2}\left(p_{i+1} - p_i\right) - T_{x,i-1/2}\left(p_i - p_{i-1}\right) \equiv \Delta_x \left(T_x \Delta_x p\right) \qquad (6.2h)$$

FORTRAN code does not use fractional indices for arrays. Thus the transmissibility coefficients are stored as $T_{x,\,i-1/2} = T_x(i),\ i = 1, \cdots, NX+1$. No flow boundaries are represented by $T_{x,1/2} = T_x(1) = 0;\quad T_{x,\,NX+1/2} = T_x(NX+1) = 0$.

The finite difference approximation for the divergence can now be expressed in the finite difference notation.

$$(\Delta x \Delta y \Delta z)\nabla \cdot \left[\frac{\mathbf{k}\, k_r\, b}{\mu}(\nabla p - \rho\, g\, \nabla D)\right] \approx \Delta\left[T\,(\Delta p - \rho\, g\, \Delta D)\right] =$$
$$+\Delta_x\left[T_x\,(\Delta_x\, p - \rho\, g\, \Delta_x\, D)\right] \qquad (6.2i)$$
$$+\Delta_y\left[T_y\,(\Delta_y\, p - \rho\, g\, \Delta_y\, D)\right]$$
$$+\Delta_z\left[T_z\,(\Delta_z\, p - \rho\, g\, \Delta_z\, D)\right]$$

This equation represents the net influx into a grid block in STB/D or Mcf/D from all of the adjacent grid blocks.

The source terms, $\tilde{q}_w, \tilde{q}_o,$ and $\tilde{q}_g$ have the units of rate per unit bulk volume of the reservoir. By multiplying the source terms by $(\Delta x \Delta y \Delta z)$, the bulk volume of the grid block, we have

$$q = (\Delta x \Delta y \Delta z)\tilde{q} \qquad (6.2j)$$

where *q* now has the units of rate of injection or negative of the rate of production (in STB/D or Mcf/D) into the grid block.

The accumulation term was expressed as a difference in time.

$$\frac{1}{\Delta t}\Delta_t(\phi\, b\, S) = \frac{\phi_{n+1}\, b_{n+1}\, S_{n+1} - \phi_n\, b_n\, S_n}{\Delta t} \qquad (6.2k)$$

The pressure dependence of the porosity can be expressed as

$$\phi_{n+1} = \phi_o\,(1 + c_r\, \delta\, p_w)_{n+1} \qquad (6.2l)$$

ϕ_o = porosity at pressure, p^o
c_r = rock compressibility
δp = difference in water phase pressure from p^o .
The porosity, ϕ_o , can be factored out of equation (6.2k).

$$\frac{1}{\Delta t}\Delta_t(\phi\, b\, S) = \frac{\phi_o}{\Delta t}\Delta_t\left[(1 + c_r\, \delta\, p_w)\, b\, S\right] \qquad (6.2m)$$

The equation for the accumulation can be multiplied by the grid block bulk volume, $\Delta x \Delta y \Delta z$, to be expressed as follows.

$$\frac{(\Delta x \Delta y \Delta z)}{\Delta t} \Delta_t (\phi b S) = \frac{V_{po}}{\Delta t} \left[(1 + c_r \delta p_w) b S \right]$$

where (6.2n)

$$V_{po} = \phi_o \Delta x \Delta y \Delta z$$

or in practical units,

$$V_{po} = \phi_o \frac{\Delta x \Delta y \Delta z}{5.615} \text{ , grid block pore volume, bbl.} \quad (6.2o)$$

The finite difference approximation to the conservation equation for water and oil can now be expressed as

$$\Delta \left[T_p \left(\Delta p_p - \rho_p g \Delta D \right) \right] + q_p = \frac{V_{po}}{\Delta t} \Delta_t \left[(1 + c_r \delta p_w) b_p S_p \right]$$

where (6.2p)

$$p = w, o$$

The conservation equation for gas with the black oil model can be expressed as

$$\Delta \left[T_g \left(\Delta p_g - \rho_g g \Delta D \right) \right] + \Delta \left[R_s T_o \left(\Delta p_o - \rho_o g \Delta D \right) \right] + q_g$$
$$= \frac{V_{po}}{\Delta t} \left\{ \Delta_t \left[(1 + c_r \delta p_w) b_g S_g \right] + \Delta_t \left[(1 + c_r \delta p_w) R_s b_o S_o \right] \right\} \quad (6.2q)$$

6.3 Variable Grid Spacing

Similar concepts apply to the formulation of the difference equations as with equal grid spacing. However, the expressions for the interblock transmissibility coefficient and grid block pore volume will be different.

With variable grid spacing, the grid system and finite difference expressions can be formulated with either the "grid block" formulation as shown on Figure 6.3 or the "grid point" formulation shown on Figure 6.4. The two formulations become the same with equal grid spacing. With the grid block formulation, the length of the grid blocks, Δx_i, Δy_j, and Δz_k are specified and the grid points are determined as the midpoint of the grid blocks. With the grid point formulation, the distances between the grid points, $\Delta x_{i+1/2}$, $\Delta y_{j+1/2}$, and $\Delta z_{k+1/2}$ are specified and the faces of the grid block are defined to be at a equal distance between grid points. The boundaries of the region is equal distance from grid point as the opposite face of the grid block.

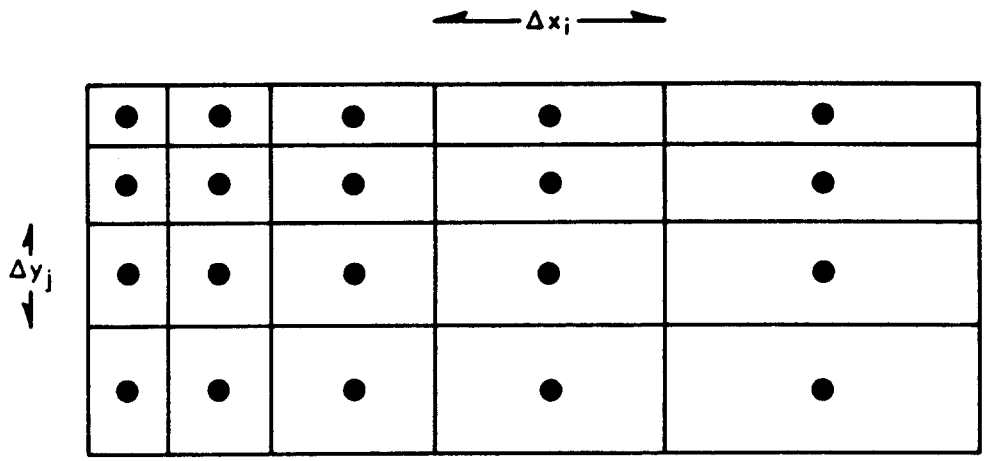

Fig. 6.3 Grid block formulation

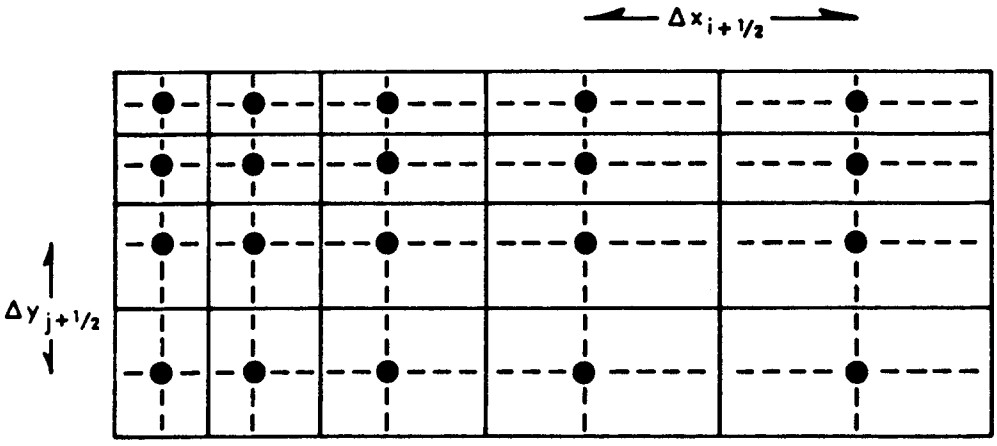

Fig. 6.4 Grid point formulation

The grid block formulation has been thought to be less accurate than the grid point formulation because the truncation error analysis requires that the grid spacing vary smoothly in space for the grid block formulation[1]. However, no significant difference in accuracy has been observed between the grid block and grid point formulation in a number of practical test problems. It is more difficult to select grid points than grid blocks for a practical reservoir problem. The points must be selected such that the grid block boundaries coincide with the reservoir external boundaries and internal boundaries such as lease boundaries or between different rock properties in stratified reservoirs.

The finite difference expression for the spatial derivatives are different between the grid block and grid point formulations. The truncation error is proportional to Δx as Δx goes to zero if the spatial distribution of Δx changes uniformly. The truncation is proportional to Δx^2 for equal grid spacing.

The finite difference expression for the spatial derivative with the **grid block formulation** is

$$\frac{\partial}{\partial x}\left(K\frac{\partial p}{\partial x}\right)_i \approx \frac{K_{i+1/2}\dfrac{\left(p_{i+1}-p_i\right)}{1/2\left(\Delta x_i+\Delta x_{i+1}\right)}-K_{i-1/2}\dfrac{\left(p_i-p_{i-1}\right)}{1/2\left(\Delta x_i+\Delta x_{i-1}\right)}}{\Delta x_i} \quad (6.3a)$$

The expression for the interblock transmissibility coefficient is

$$T_{x,i+1/2} = K_{i+1/2}\frac{\Delta y_j\,\Delta z_k}{1/2\left(\Delta x_i+\Delta x_{i+1}\right)} \quad (6.3b)$$

where the average permeability is computed as

$$\bar{k}_{i+1/2} = \frac{k_i\,k_{i+1}\left(\Delta x_i+\Delta x_{i+1}\right)}{k_{i+1}\Delta x_i + k_i\,\Delta x_{i+1}} \quad (6.3c)$$

The grid block pore volume is expressed as

$$V_{po} = \phi_o\,\Delta x_i\,\Delta y_j\,\Delta z_k \quad (6.3d)$$

The spatial derivative is approximated with the **grid point formulation** as

$$\frac{\partial}{\partial x}\left(K\frac{\partial p}{\partial x}\right)_i \approx \frac{K_{i+1/2}\dfrac{\left(p_{i+1}-p_i\right)}{\Delta x_{i+1/2}} - K_{i-1/2}\dfrac{\left(p_i-p_{i-1}\right)}{\Delta x_{i-1/2}}}{1/2\left(\Delta x_{i+1/2}+\Delta x_{i-1/2}\right)} \tag{6.3e}$$

The expression for the interblock transmissibility coefficient is

$$T_{x,i+1/2} = K_{i+1/2}\frac{1/2\left(\Delta y_{j+1/2}+\Delta y_{j-1/2}\right)1/2\left(\Delta z_{k+1/2}+\Delta z_{k-1/2}\right)}{\Delta x_{i+1/2}} \tag{6.3f}$$

where the average permeability is computed as

$$\bar{k}_{i+1/2} = \frac{2k_i k_{i+1}}{k_{i+1}+k_i} \tag{6.3g}$$

The expression for the grid block pore volume is

$$V_{po} = \phi_o \; 1/2\left(\Delta x_{i+1/2}+\Delta x_{i-1/2}\right)1/2\left(\Delta y_{j+1/2}+\Delta y_{j-1/2}\right)1/2\left(\Delta z_{k+1/2}+\Delta z_{k-1/2}\right) \tag{6.3h}$$

6.4 Material Balance Check

The conservation equations were derived based on the conservation of matter. The finite difference equations should also be formulated so that it too is consistent with the conservation of matter. It will be shown that the finite difference equations that have been formulated in the previous sections will satisfy the material balance condition,

(net cumulative injection) = (present fluid in place) - (initial fluid in place) (6.4a)

where net cumulative injection is the difference between cumulative injection and cumulative production. This material balance condition should be satisfied exactly at any time during the simulation if the finite difference equations are computed with no error. However, the material balance will not be satisfied exactly because of the round off error of the computer and because of using only a finite number of iterations in the pressure solution technique. In principal the error due to using a finite number of iterations can be reduced to the level of the round off error by reducing the magnitude of the error tolerance for the iterations. The user must judge between the accuracy of the material balance and the computing cost of taking a large number of iterations.

The total material balance is derived by summing the finite difference conservation equations over all grid blocks and over all time steps that have been computed. Denote the summation of one of the conservation equations as

$$\sum_{i,j,k}\sum_{n=0}^{N_t-1}\Delta t_{n+1}\left\{\Delta\left[T\left(\Delta p-\rho g\,\Delta D\right)\right]+q-\frac{V_{po}}{\Delta t_{n+1}}\Delta_t\left[\left(1+c_r\,\delta p_w\right)b\,S\right]\right\}=0 \qquad (6.4b)$$

where

$$\sum_{i,j,k}$$

denotes the summation over all grid blocks and

$$\sum_{n=0}^{N_t-1}$$

denotes the summation from the initial conditions to time, t_{Nt}. The two summations are interchangeable.

It can be shown that the summation of the flux terms over all the grid blocks is equal to zero. To illustrate this, the flux in the x-direction will be considered. To simplify the notation, define $\Delta\Phi$ such that

$$\Delta_x T\left(\Delta_x p-\rho g\,\Delta_x D\right)=\Delta_x T\Delta_x\Phi \qquad (6.4c)$$

The summation from the first, *i=1*, to the last, *i=NX* grid block in the x-direction is

$$\begin{aligned}\sum_{i=1}^{NX}\Delta_x T\Delta_x\Phi = {} & T_{3/2}\Delta\Phi_{3/2}-T_{1/2}\Delta\Phi_{1/2}\\ & +T_{5/2}\Delta\Phi_{5/2}-T_{3/2}\Delta\Phi_{3/2}\\ & \cdots\\ & +T_{i-1/2}\Delta\Phi_{i-1/2}-T_{i-3/2}\Delta\Phi_{i-3/2}\\ & +T_{i+1/2}\Delta\Phi_{i+1/2}-T_{i-1/2}\Delta\Phi_{i-1/2}\\ & \cdots\\ & +T_{NX+1/2}\Delta\Phi_{NX+1/2}-T_{NX-1/2}\Delta\Phi_{NX-1/2}\end{aligned} \qquad (6.4d)$$

All of the terms cancel by pairs except the first and the last.

$$\begin{aligned}\sum_{i=1}^{NX}\Delta_x T\Delta_x\Phi &= T_{NX+1/2}\Delta\Phi_{NX+1/2}-T_{-1/2}\Delta\Phi_{-1/2}\\ &= 0\end{aligned} \qquad (6.4e)$$

The transmissibility coefficient at the boundaries, $T_{-1/2}$ and $T_{NX+1/2}$ are zero so the entire sum is zero. The same applies to the y and z directions. This condition that the sum of the flux terms are zero results from formulating the flux terms such that the flux out of one grid block is equal to the flux into the adjacent grid block.

Denote the cumulative injection or the negative of the cumulative production of a well by Q.

$$Q = \sum_{n=0}^{N_t-1} q\,\Delta t_{n+1} \tag{6.4f}$$

The net cumulative injection into all of the wells in the reservoir is

$$\sum_{i,j,k} Q = \sum_{i,j,k} \sum_{n=0}^{N_t-1} q\,\Delta t_{n+1} \tag{6.4g}$$

To simplify the notation, denote the accumulation terms as

$$\begin{aligned} \frac{V_{po}}{\Delta t}\Delta_t\left[\left(1+c_r\,\delta p_w\right) b\,S\right] &= \frac{V_{po}}{\Delta t}\Delta_t C \\ &= \frac{V_{po}}{\Delta t}\left(C_{n+1}-C_n\right) \end{aligned} \tag{6.4h}$$

The summation of the accumulation terms over time can be expressed as

$$\begin{aligned} &\sum_{n=0}^{N_t-1} \Delta t_{n+1}\frac{V_{po}}{\Delta t_{n+1}}\left(C_{n+1}-C_n\right) \\ &= V_{po}\begin{bmatrix}\left(C_1-C_0\right)+\left(C_2-C_1\right)+\ldots+\left(C_n-C_{n-1}\right)+\left(C_{n+1}-C_n\right)+\ldots \\ +\left(C_{N_t-1}-C_{N_t-2}\right)+\left(C_{N_t}-C_{N_t-1}\right)\end{bmatrix} \\ &= V_{po}\left(C_{N_t}-C_0\right) \end{aligned} \tag{6.4i}$$

The particular expansion of the accumulation term when expressing in finite difference form is now evident. It had to be expanded as an identity for the summation over the time steps to cancel pairwise.

The summation of the accumulation terms over the grid blocks and over time is

$$\sum_{i,j,k}\sum_{n=0}^{N_t-1}\Delta t_{n+1}\frac{V_{po}}{\Delta t_{n+1}}\Delta_t\left[\left(1+c_r\,\delta p_w\right)b\,S\right]$$
$$=\sum_{i,j,k}V_{po}\left[\left(1+c_r\,\delta p_w\right)b\,S\right]_{N_t}-\sum_{i,j,k}V_{po}\left[\left(1+c_r\,\delta p_w\right)b\,S\right]_0 \qquad (6.4j)$$
$$=PFIP-OFIP$$

where *PFIP* is present fluid in place and *OFIP* is original fluid in place.

Substituting the earlier equations into the above equation, we have

$$\sum_{i,j,k}Q-PFIP+OFIP=0 \qquad (6.4k)$$

This is an expression for the total material balance. It can also be expressed as

$$\frac{PFIP-\sum Q}{OFIP}=1$$
$$\frac{PFIP-OFIP}{\sum Q}=1 \qquad (6.4l)$$
$$\frac{OFIP+\sum Q}{PFIP}=1$$

The values of *OFIP*, *PFIP*, and ΣQ can be computed during each simulation run and the value of the ratios examined. The deviation of the ratios from 1.0 is a measure of the material balance error as a fraction of the quantity in the denominator. The user needs to compare the magnitudes of *OFIP* and ΣQ to determine which ratio is a more significant measure of the error. For example, the equation with the cumulative production in the denominator should *not* be used as a measure of the error in the water material balance if little or no water is produced.

6.5 Solution of the Difference Equations

The finite difference equations discussed in the previous sections can be solved by two different procedures for multiphase problems and with a simplification for the single phase problems. The multiphase problems can either be solved with the implicit in pressure, explicit in saturation (IMPES) procedure or the procedure that is implicit in both pressure and saturations. Most reservoir recovery processes can be solved using the IMPES procedure. However, in single well coning problems with very small grid blocks near the well, it is necessary to use the totally implicit procedure. In single phase problems, the pressure or the real gas pseudo pressure is the only dependent variable and is computed implicitly. Thus, there is no time step limitation due to stability for single phase problems.

6.6 The Implicit Pressure - Explicit Saturation Procedure

The IMPES procedure is implicit in pressure and explicit in saturation. All of the capillary pressure and relative permeability terms are computed at the old time level, t_n , using the known value of saturation, S_n. The only unknown saturation terms in each finite difference equation is in the time difference term. Thus, the procedure is a forward difference or explicit procedure for the saturations. The flux terms are formulated with the unknown pressures, p_{n+1}, at the new time level, t_{n+1}. Thus, the procedure for the pressure is a backward difference or implicit procedure.

The basis of the procedure is to combine the conservation equations such as to eliminate the $\Delta_t S$ terms that contain the unknown saturations. The result is then a single equation for the oil phase pressure. This equation is solved for the pressure using one of the methods such as alternating direction implicit procedure (ADIP), strongly implicit procedure (SIP), conjugate gradient, or Gaussian elimination. After the pressures are computed, the saturations are explicitly computed from their respective conservation equation.

The accumulation terms are expanded such that the time differences can be expressed in terms of the time difference in pressure, $\Delta_t P$, and time difference in saturations, $\Delta_t S$. The equations are then combined to eliminate the $\Delta_t S$ terms. The expansion should be expressed such that S_{n+1} does not appear in any of the coefficients after the $\Delta_t S$ terms are eliminated. Moreover, the expansion should be consistent with the material balance.

A consistent expansion of the time difference of product *abc* in terms of the time difference of each of the factors is as follows.

$$\begin{aligned}\Delta_t(abc) &= a_{n+1}b_{n+1}c_{n+1} - a_n b_n c_n - a_{n+1}b_{n+1}c_n + a_{n+1}b_{n+1}c_n + a_{n+1}b_n c_n - a_{n+1}b_n c_n \\ &= a_{n+1}b_{n+1}(c_{n+1}-c_n) + a_{n+1}c_n(b_{n+1}-b_n) + b_n c_n(a_{n+1}-a_n) \\ &= a_{n+1}b_{n+1}\Delta_t c + a_{n+1}c_n\Delta_t b + b_n c_n \Delta_t a\end{aligned} \tag{6.6a}$$

Note that the coefficient, c_{n+1} does not appear in any of the terms. The expansion of the accumulation terms in this way is consistent with the material balance discussed earlier.

The expansion of the accumulation terms for the water and oil conservation equations are as follows.

$$\Delta_t\left[(1+c_r\,\delta p_w)bS\right] = (1+c_r\,\delta p_w)_{n+1}b_{n+1}\Delta_t S + \left[(1+c_r\,\delta p_w)_{n+1}b'S_n + c_r b_n S_n\right]\Delta_t p \tag{66b}$$

where

$$b' = \frac{b_{n+1}-b_n}{p_{n+1}-p_n} \tag{6.6c}$$

$$\Delta_t p_w = \Delta_t p \tag{6.6d}$$

The unsubscripted p is used to denote the oil phase pressure, p_o . The term, b' contains the unknowns, b_{n+1} and p_{n+1}. It is estimated by using the last iterated value for b_{n+1} and p_{n+1}. Equation (6.6d) implies that the time change in capillary pressure is neglected.

The expansion of the accumulation terms for the gas conservation equations is as follows.

$$\begin{aligned}&\Delta_t\left[(1+c_r\,\delta p_w)(b_g S_g + b_o R_s S_o)\right] \\ &= (1+c_r\,\delta p_w)_{n+1}b_{g,n+1}\Delta_t S_g + (1+c_r\,\delta p_w)_{n+1}(b_o R_s)_{n+1}\Delta_t S_o \\ &+ \left\{(1+c_r\,\delta p_w)_{n+1}\left[b'_g S_{g,n} + (b_o R_s)' S_{o,n}\right] + c_r(b_g S_g + b_o R_s S_o)_n\right\}\Delta_t p\end{aligned} \tag{6.6e}$$

where

$$b'_g = \frac{b_{g,n+1}-b_{g,n}}{p_{n+1}-p_n} \tag{6.6f}$$

$$(b_o R_s)' = \frac{(b_o R_s)_{n+1} - (b_o R_s)_n}{p_{n+1}-p_n} \tag{6.6g}$$

The accumulation terms for the conservation equations can now be expressed as

water: $$\frac{V_{po}}{\Delta t}\Delta_t\left[\left(1+c_r\,\delta p_w\right)b_w\,S_w\right]=z_{10}\,\Delta_t p+z_{11}\,\Delta_t S_w \qquad (6.6h)$$

oil: $$\frac{V_{po}}{\Delta t}\Delta_t\left[\left(1+c_r\,\delta p_w\right)b_o\,S_o\right]=z_{20}\,\Delta_t p+z_{22}\,\Delta_t S_o \qquad (6.6i)$$

gas: $$\frac{V_{po}}{\Delta t}\Delta_t\left[\left(1+c_r\,\delta p_w\right)\left(b_g\,S_g+b_o\,R_s\,S_o\right)\right]=z_{30}\,\Delta_t p+z_{33}\,\Delta_t S_g+z_{32}\,\Delta_t S_o$$

(6.6j)

The first subscript of z denotes the number (1 for water, 2 for oil, 3 for gas) of the conservation equation and the second subscript denotes the time difference term (0 for pressure, 1 for water, 2 for oil, 3 for gas).

The coefficients, z_{ij} are as follows:

$$z_{10}=\frac{V_{po}}{\Delta t}\left[\left(1+c_r\,\delta p_w\right)_{n+1}b'_w\,S_{w,n}+c_r\,b_{w,n}\,S_{w,n}\right]$$

(6.6k)

$$z_{11}=\frac{V_{po}}{\Delta t}\left(1+c_r\,\delta p_w\right)_{n+1}b_{w,n+1} \qquad (6.6l)$$

$$z_{20}=\frac{V_{po}}{\Delta t}\left[\left(1+c_r\,\delta p_w\right)_{n+1}b'_o\,S_{o,n}+c_r\,b_{o,n}\,S_{o,n}\right]$$

(6.6m)

$$z_{22}=\frac{V_{po}}{\Delta t}\left(1+c_r\,\delta p_w\right)_{n+1}b_{o,n+1} \qquad (6.6n)$$

$$z_{30}=\frac{V_{po}}{\Delta t}\left\{\left(1+c_r\,\delta p_w\right)_{n+1}\left[b'_g\,S_{g,n}+\left(b_o\,R_s\right)'S_{o,n}\right]+c_r\left(b_g\,S_g+b_o\,R_s\,S_o\right)_n\right\} \qquad (6.6o)$$

$$z_{32}=\frac{V_{po}}{\Delta t}\left(1+c_r\,\delta p_w\right)_{n+1}\left(b_o\,R_s\right)_{n+1} \qquad (6.6p)$$

$$z_{33}=\frac{V_{po}}{\Delta t}\left(1+c_r\,\delta p_w\right)_{n+1}b_{g,n+1} \qquad (6.6q)$$

It should be noted that although the coefficients are a function of the unknown pressure, p_{n+1} they are not a function of the unknown saturations, S_{n+1}. Furthermore, the coefficients have been defined such that it is consistent with the material balance.

The conservation equations can now be expressed as

$$\Delta\left[T_w\left(\Delta p_w - \rho_w\, g\, \Delta D\right)\right] + q_w = z_{10}\,\Delta_t p + z_{11}\,\Delta_t S_w \tag{6.6r}$$

$$\Delta\left[T_o\left(\Delta p_o - \rho_o\, g\, \Delta D\right)\right] + q_o = z_{20}\,\Delta_t p + z_{22}\,\Delta_t S_o \tag{6.6s}$$

$$\begin{aligned}&\Delta\left[T_g\left(\Delta p_g - \rho_g\, g\, \Delta D\right)\right] + \Delta\left[R_s\, T_o\left(\Delta p_o - \rho_o\, g\, \Delta D\right)\right] + q_g \\ &= z_{30}\,\Delta_t p + z_{32}\,\Delta_t S_o + z_{33}\,\Delta_t S_g\end{aligned} \quad . \tag{6.6t}$$

The conservation equations will now be combined to eliminate the $\Delta_t S$ terms. Multiplication of equations (6.6r), (6.6s), and (6.6t)by the factors a_1, a_2, and a_3, respectively, and adding results in

$$\begin{aligned}&a_1\,\Delta\left[T_w\left(\Delta p_w - \rho_w\, g\, \Delta D\right)\right] + a_2\,\Delta\left[T_o\left(\Delta p_o - \rho_o\, g\, \Delta D\right)\right] \\ &a_3\,\Delta\left[T_g\left(\Delta p_g - \rho_g\, g\, \Delta D\right)\right] + a_3\,\Delta\left[R_s\, T_o\left(\Delta p_o - \rho_o\, g\, \Delta D\right)\right] + q \\ &= C\,\Delta_t p + a_1\, z_{11}\,\Delta_t S_w + a_2\, z_{22}\,\Delta_t S_o + a_3\, z_{33}\,\Delta_t S_g + a_3\, z_{32}\,\Delta_t S_o\end{aligned} \tag{6.6u}$$

where

$$q = a_1\, q_w + a_2\, q_o + a_3\, q_g \tag{6.6v}$$

$$C = a_1\, z_{10} + a_2\, z_{20} + a_3\, z_{30} \tag{6.6w}$$

The condition that

$$S_w + S_o + S_g = 1 \tag{6.6x}$$

imposes the condition that

$$\Delta_t S_w + \Delta_t S_o + \Delta_t S_g = 0 \quad . \tag{6.6y}$$

We wish to impose the condition that the $\Delta_t S$ terms in equation (6.6u) disappear.

$$a_1\, z_{11}\,\Delta_t S_w + (a_2\, z_{22} + a_3\, z_{32})\Delta_t S_o + a_3\, z_{33}\,\Delta_t S_g = 0 \tag{6.6z}$$

Equation (6.6z) will be satisfied if the coefficients of the $\Delta_t S$ terms are equal. This imposes the following conditions on the a_1, a_2, and a_3 factors.

$$a_1 = \frac{V_{po}}{z_{11}\,\Delta t} \tag{6.6aa}$$

$$a_2 = \frac{V_{po}}{z_{22}\,\Delta t}\left(1 - \frac{z_{32}}{z_{33}}\right) \tag{6.6bb}$$

$$a_3 = \frac{V_{po}}{z_{33}\,\Delta t} \tag{6.6cc}$$

The factors defined in this way have a magnitude approximately equal to the formation volume factor.

When the free gas phase is not present in a grid block, the gas conservation equation is redundant with the oil equation. In this case only the water and oil conservation equations are considered and the factors are

$$a_1 = \frac{V_{po}}{z_{11}\,\Delta t} \tag{6.6dd}$$

$$a_2 = \frac{V_{po}}{z_{22}\,\Delta t} \quad . \tag{6.6ee}$$

The flux terms are expressed in terms of p_w , p_o , and p_g. They need to be expressed in terms of the oil phase pressure at the new time level, p_{n+1}. The capillary pressure will be expressed using the saturations at the old time level.

$$p_{w,n+1} = p_{n+1} - P_{cwo}\left(S_{w,n}\right) \tag{6.6ff}$$

$$p_{o,n+1} = p_{n+1} \tag{6.6gg}$$

$$p_{g,n+1} = p_{n+1} + P_{cgo}\left(S_{g,n}\right) \tag{6.6hh}$$

Equation (6.6u) can now be expressed as

$$a_1\,\Delta\left(T_w \Delta p_{n+1}\right) + a_2\,\Delta\left(T_o \Delta p_{n+1}\right) + a_3\left[\Delta\left(T_g \Delta p_{n+1}\right) + \Delta\left(R_s\, T_o \Delta p_{n+1}\right)\right] + B_1 + q = C\,\Delta_t p \tag{6.6ii}$$

where

$$B_1 = -a_1 \Delta\left(T_w \Delta P_{cwo}\right) + a_3 \Delta\left(T_g \Delta P_{cgo}\right)$$
$$-a_1 \Delta\left(T_w \rho_w g \Delta D\right) - a_2 \Delta\left(T_o \rho_o g \Delta D\right)$$
$$-a_3\left[\Delta\left(T_g \rho_g g \Delta D\right) + \Delta\left(R_s T_o \rho_o g \Delta D\right)\right]$$

(6.6jj)

The term, B_1 represents the flux due to the capillary pressure and gravity. It changes during the iterative solution for p_{n+1} only through changes in the value of a_1, a_2, and a_3. The flux terms can be further summarized as

$$\Delta\bar{T}\Delta p_{n+1} = a_1 \Delta\left(T_w \Delta p_{n+1}\right) + a_2 \Delta\left(T_o \Delta p_{n+1}\right)$$
$$+a_3\left[\Delta\left(T_g \Delta p_{n+1}\right) + \Delta\left(R_s T_o \Delta p_{n+1}\right)\right]$$

(6.6kk)

Equation (6.6kk) is only a notation simplification and does not imply that there are no factors multiplying the spatial difference operators. Equation (6.6ii) can be written with this notation as

$$\Delta\bar{T}\Delta p_{n+1} + B_1 + q = C\,\Delta_t p \qquad \text{(6.6ll)}$$

Equation (6.6ll) can be written in residual form to reduce the effect of the computer round off error. Denote the change in p from the k^{th} iteration to the $k+1^{\text{th}}$ iteration as

$$\delta p^{k+1} = p_{n+1}^{k+1} - p_{n+1}^{k} \qquad \text{(6.6mm)}$$

where p^k is the k^{th} iterate to p_{n+1}. Equation (6.6ll) can now be rewritten as

$$\Delta\bar{T}\Delta\delta p^{k+1} - C\,\delta p^{k+1} = B^k \qquad . \qquad \text{(6.6nn)}$$

where

$$B^k = -\Delta\bar{T}\Delta p^k - B_1 - q + C\left(p^k - p_n\right) \qquad \text{(6.6oo)}$$

The term B^k represents the residual after the k^{th} iteration. The value of B^k is zero when p^k satisfies equation (6.6ll). Equation (6.6nn) can be solved with procedures such as ADIP, SIP, conjugate gradient, or Gaussian elimination. The iterated value of the estimate to p_{n+1} is

$$p_{n+1}^{k+1} = p_n^k + \delta p^{k+1}$$

or (6.6pp)

$$DP^{k+1} = DP^k + \delta p^{k+1}$$

where

$$DP^k = p_{n+1}^k - p_n$$
$$p^0 = p_n \quad \text{(6.6qq)}$$

The values of a_i and z_{ij} are dependent on pressure and thus they are updated on each iteration of the pressure equation.

The iterations are continued until the residuals become small enough. The magnitude of the value of the residual can be measured in several ways.

$$ABSUM = \sum_{i,j,k} \left| B^k \right| \quad \text{(6.6rr)}$$

$$ALGSUM = \sum_{i,j,k} B^k \quad \text{(6.6ss)}$$

$$BXMAX = \max_{i,j,k} \left| B^k \right| \quad \text{(6.6tt)}$$

$$BXVMAX = \max_{i,j,k} \left(\frac{\left| B^k \right|}{V_{po}} \right) \quad \text{(6.6uu)}$$

where the summation or maximum value are evaluated over all grid blocks. The units of B^k are barrels (at reservoir conditions) per day. Thus, the value of B^k is a measure of the error in barrels per day remaining after k iterations.

The value of the sum of residuals is often normalized by the total pore volume, *TPV*.

$$TPV = \sum_{i,j,k} V_{po} \quad \text{(6.6ww)}$$

The measure of the error that is compared with the closure tolerance, TOL, for Cartesian or reservoir curvilinear coordinates is

$$\frac{ABSUM}{TPV} \le TOL \quad \text{(6.6xx)}$$

In radial coordinates, *BXVMAX* is compared with *TOL*. The residuals that are printed out on each iteration by option are *ABSUM/TPV*, *ALGSUM/TPV*, and *BXVMAX*. The user should specify the value of *TOL* such that the final value of *ABSUM* or *BXVMAX* represent a tolerable error, e.g., a small fraction (0.005 to 0.05) of the total production rate. If the value of *TOL* is too small, a large number of iterations (and large computing time) may be necessary to reach the specified accuracy. On the other hand, if the value of *TOL* is too large, material balance errors may become significant.

After the solution for the pressure has converged, the values of $\Delta_t S_w$ and $\Delta_t S_o$ are computed from equations (6.6r) and (6.6s). $\Delta_t S_g$ is not computed from equation (6.6t) because $\Delta_t S_g$ is dependent on $\Delta_t S_w$ and $\Delta_t S_o$ from equation (6.6y). The gas conservation equation, (6.6t) is represented in the combined equation for pressure, equation (6.6ll). If the free gas phase is not present, only one of either $\Delta_t S_w$ or $\Delta_t S_o$ needs to be computed.

The limitation of the explicit calculation of the saturations is that the size of the time step, Δt, may be limited by stability[3]. The time step limitation due to evaluating the relative permeability at the old, t_n , time step can be approximated as

$$\Delta t \le \frac{V_{po}}{f'|u\,A|} \qquad \text{(6.6yy)}$$

where *uA* represents the volumetric flow rate through a grid block (throughput) and *f'* is the derivative of the fractional flow curve. The time step limitation may be severe with small grid blocks (V_{po} small) or with large throughput ($|u\,A|$ large). This condition occurs in coning problems. Large value of *f'* can become significant for unfavorable mobility ratio displacement, but is not a severe limitation for favorable mobility ratio displacement.

The time step limitation due to evaluating the capillary pressure at the old time step can be approximated as

$$\Delta t \le \frac{V_{po}}{P_c'\left(T_x + T_y + T_z\right)} \qquad \text{(6.6zz)}$$

where P_c' is the derivative of the capillary pressure curve and T_X , T_Y , T_Z are the interblock transmissibility coefficients. The time step limitation may be severe with a small grid block, large capillary pressure, or large transmissibility (due to large permeability). For this reason the nearly vertical section of the P_C curve near S_{WC} should be modified to be an extrapolation of the remainder of the curve.

Referrences

1. Settari, A. and Aziz, K.: "Use of Irregular Grid in Reservoir Simulation", SPEJ, April, 1972, 103-114.
2. Weiser, A. and Wheeler, M. F.: "On Convergence of Block-entered Finite Difference for Elliptic Problems", SIAM J. Numer. Anal., Vol. 24, No. 2, April 1988, 351-375.
3. Todd, M.R., O'Dell, P.M., and Hirasaki, G.J.: "Methods for Increased Accuracy in Numerical Reservoir Simulators", Soc. Pet. Eng. J., (December, 1972), 515-539.

Assignment 6.1 1-D, 1-phase, constant compressibility simulation for pressure. Parabolic PDE; Recovery by pressure depletion

This assignment is a step toward 2,3-D, 2-phase IMPES simulation in heterogeneous systems of different geometry and multiple wells. Write the code in anticipation that additional features will be added later.

$$\nabla p_o - \rho_o g \nabla D = \nabla \Phi_o \Rightarrow \nabla p$$

$$b_o = b_o\left(@\ p_{STP}\right)\left[1 + c_o\left(p\right)\right]$$
$$\phi = \phi\left(@\ p^{ic}\right)\left[1 + c_r\left(p - p^{ic}\right)\right]$$

$$p_{n+1}^{k} - p_n \Rightarrow DP$$
$$p_{n+1}^{0} - p_n = p_n - p_n = 0$$
$$p_{n+1}^{k+1} - p_{n+1}^{k} = \delta p^{k+1} \Rightarrow DPK$$
$$p_{n+1}^{k+1} - p_n \Rightarrow DP + DPK$$
$$p_{n+1}^{k+1} = p_n + DP$$
when converged,
$$p_{n+1} = p_n + DP$$

LX=1,000 feet, NX is specified. LY=100 feet, NY=1. LZ=10 feet, NZ=1.
ϕ=0.30 (@ p^{ic}), k=50 md, k_{ro}=1.0, μ=1.0 cp, p^{ic}=1,000 psi, S_o=1.
b_o(@p^{ic})=1.0, c_o=1.0E-5 1/psi, c_r=1.0E-5 1/psi.
q_o=-10 STB/day, at i=NX.
ABSUM/TPV<TOL=1.E-8 (also try TOL = 20.0 and 1.E-5. Number of iterations?)
(POIP-CUM)/OOIP must be 1.0000 to this many significant digits.
Case 1: NX=10, DPMAX=10.0 psi
Case 2: NX=40, DPMAX=2.0 psi

Compare profiles and history with analytical solution using program linear.*m* found in owlnet, *gjh, /class/linear.m*.
Show your code.

Analytical solution for assignment 6.1

$$p_{avg}(t) = p^{ic} + \frac{5.615\,q\,t}{\phi\,c\,L_x\,dy\,dz}$$

$$p_{ss}(x) = \frac{887.2\,q\,\mu\,L_x}{dy\,dz\,k\,k_r^o\,b}\left(\frac{3\left(x/L_x\right)^2 - 1}{6}\right)$$

$$p(t,x) = p_{avg}(t) + p_{ss}(x) - \frac{887.2\,q\,\mu\,L_x}{dy\,dz\,k\,k_r^o\,b}\left(\frac{2}{\pi^2}\right)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\exp\left(-\kappa\left(\pi\,n/L_x\right)^2 t\right)\cos\left(\pi\,n\,x/L_x\right)$$

where

$$\kappa = \frac{0.00633\,k\,k_r^o}{\phi\,c\,\mu}$$

Assignment 6.2 Variable grid spacing

Do assignment 6.1 except use NX=10 and DPMAX=2.0 psi with variable grid spacing. Let DX = 140. 140. 140. 140. 140. 100. 80. 60. 40. 20..

Assignment 6.3 Extension to two dimensions

Add option for discretization in the y direction to the code of assignment 6.1. At this point, do the matrix inversion only in the x OR y direction. Verify that the same result is obtained when NX=1, LX=100., NY=10, LY=1000. as for NX=10, LX=1000. , NY=1, LY=100.