Chapter 4 1-Dimensional Displacement with Pressure and Capillary Pressure

The formulation with saturation as the only dependent variable can be used for 1-D problems but can not be extended to two or three dimensions. The total flux is no longer known apriori in more than one dimension. One could calculate the total flux field and calculate stream tubes in which the fluid flows as if the displacement is one dimensional within the stream tube. This approach was widely used before finite difference simulation came into practice. This approach is valid if the total flux field does not change with time. However, the total flux changes if relative well rates change or the displacement is not a unit mobility ratio displacement.

In 1-D it is not necessary to include pressure as a dependent variable. However, we do so in anticipation of multi-dimensional problems. We first start with the Buckley-Leverett displacement problem that we calculated with the method of characteristics and with finite difference with saturation as the only dependent variable. The capillary pressure will be included. The first finite difference formulation is the implicit pressure, explicit saturation (**IMPES**) formulation. This will require inversion of a tridiagonal matrix. This formulation will now have an additional stability criterion because of the introduction of capillary pressure. We will overcome the stability limitations by using a **semi-implicit** formulation. This will require inversion of a block tridiagonal matrix.

Differential Equations and boundary Conditions

The formulation will use the subscript *d* and *i* to denote the displaced and invading phases rather than oil and water. The capillary pressure will be specified such that the invading phase is nonwetting (i.e. drainage) and the capillary pressure is an increasing function of the invading phase saturation when the capillary pressure coefficient is positive. By making this coefficient negative, the invading phase will be the wetting phase (i.e., imbibition) and the capillary pressure will decrease in magnitude with increase in the invading phase saturation.

$$u_d = -\frac{kk_{rd}(S)}{\mu_d}\frac{\partial p_d}{\partial x}$$

$$u_i = -\frac{kk_{ri}(S)}{\mu_i}\frac{\partial p_i}{\partial x}$$

$$\phi \frac{\partial S_d}{\partial t} + \frac{\partial u_d}{\partial x} = 0$$

$$\phi \frac{\partial S_i}{\partial t} + \frac{\partial u_i}{\partial x} = 0$$

$$S_d + S_i = 1.0$$

$$p_i - p_d = P_c(S)$$

$$S_i(x,0) = S_i^{IC}$$

$$x = 0: \quad u_d = 0, \quad u_i = u^{BC}$$

$$x = L: \quad (u_i = 0 \text{ or } P_c = 0), \quad p_i = p_i^{BC}$$

The boundary conditions assume that counter-current flow does not occur at either end. Different boundary conditions could be used to simulate counter-current flow.

Make variables dimensionless or normalized:

$$x_D = \frac{x}{L}, \qquad t_D = \frac{u^{BC} t}{\phi L(1 - S_{ir} - S_{dr})}$$

$$S_D = \frac{S_i - S_{ir}}{(1 - S_{ir} - S_{dr})}$$

$$k_{riD} = \frac{k_{ri}(S)}{k_{ri}^o}, \qquad k_{rdD} = \frac{k_{rd}(S)}{k_{rd}^o}$$

$$M = \frac{k_{ri}^o}{k_{rd}^o} \frac{\mu_d}{\mu_i}$$

Determine the expression for the dimensionless pressure by specifying a characteristic pressure that eliminates the parameters from the inflow boundary condition.

$$p_D = \frac{p_i}{p^*}$$

$$u^{BC} = -\frac{k\, k_{ri}^o}{\mu_i} \frac{p^*}{L} \frac{\partial p_D}{\partial x_D}$$

$$p^* = \frac{u^{BC} \mu_i L}{k\, k_{ri}^o}$$

$$p_{iD} = \frac{k\, k_{ri}^o}{u^{BC} \mu_i L}\left(p_i - p_i^{BC}\right)$$

$$P_c(S) = \frac{\sigma}{\sqrt{k/\phi}} J(S)$$

$$p_i - p_d = P_c \quad \Rightarrow$$

$$p_{iD} - p_{dD} = \frac{\frac{\sigma}{\sqrt{k/\phi}}}{p^*} J(S)$$

$$= N_{P_c} J(S)$$

$$= P_{cD}(S)$$

$$N_{P_c} = \frac{k\, k_{ri}^o}{u^{BC} \mu_i L} \frac{\sigma}{\sqrt{k/\phi}} = \frac{\sigma\, k_{ri}^o \sqrt{\phi k}}{u^{BC} \mu_i L}$$

The capillary pressure dimensionless number is a ratio of capillary to viscous forces in a sample of finite length. It is also identified as an "end-effect" number to quantify the effect of the outflow end boundary condition (Mohanty and Miller, 1991). Notice that it is inversely proportional to the flux and system length. The dimensionless capillary pressure is expressed as the product of the capillary pressure dimensionless number and the J function.

Let $\Phi_D = p_{iD}$. Everything will now be dimensionless. Drop the subscript, *D*. The differential equations and boundary conditions are now as follows.

$$\frac{\partial S_d}{\partial t} - \frac{\partial}{\partial x}\left[\frac{k_{rd}(S)}{M}\frac{\partial\left(\Phi - P_c(S)\right)}{\partial x}\right] = 0$$

$$\frac{\partial S_i}{\partial t} - \frac{\partial}{\partial x}\left[k_{ri}(S)\frac{\partial \Phi}{\partial x}\right] = 0$$

$$S_d + S_i = 1.0$$

$$S_i(x,0) = \frac{S_i^{IC} - S_{ir}}{1 - S_{dr} - S_{ir}}$$

$$x = 0: \quad u_d = 0, \quad u_i = 1.0 = -\frac{\partial \Phi}{\partial x}(0,t)$$

$$x = 1: \quad \left(u_i = 0 \text{ OR } P_c = 0\right), \quad \Phi = 0$$

There are two dependent variables, a saturation and a potential, and two conservation equations. If we are to solve the system with an IMPES formulation, we need to reformulate the equations. The two conservation

equations are added together and the accumulation terms cancel out. The result is an elliptic equation for the potential. This combined with the conservation equation for the invading phase defines the equations to be solved by the IMPES formulation.

$$\frac{\partial}{\partial x}\left[\left(k_{ri}+\frac{k_{rd}}{M}\right)\frac{\partial \Phi}{\partial x}-\frac{k_{rd}}{M}\frac{\partial P_c}{\partial x}\right]=0$$

$$\frac{\partial S_i}{\partial t}-\frac{\partial}{\partial x}\left[k_{ri}(S)\frac{\partial \Phi}{\partial x}\right]=0$$

IMPES Finite Difference Formulation

The conservation equations are expressed in finite difference assuming equal grid spacing. The divergence operator is expressed as a central difference between the fluxes at *i+1/2* and *i-1/2*, i.e., at the bounding "faces" of the grid blocks. The saturations are not known at fractional grid locations so the value is approximated by the upstream value. A central difference for the gradient operator at these fractional grid locations results in evaluation of the potential and capillary pressures at the grid block locations. A forward difference is used for the time differential.

Let $\lambda = k_{ri}+\frac{k_{rd}}{M}$.

$$\frac{\Delta_x}{\Delta x}\left(\lambda\frac{\Delta_x \Phi}{\Delta x}\right)=\frac{\lambda_{i+1/2}\left(\Phi_{i+1}-\Phi_i\right)-\lambda_{i-1/2}\left(\Phi_i-\Phi_{i-1}\right)}{\Delta x^2}$$

$$\frac{\Delta_x}{\Delta x}\left(k_r\frac{\Delta_x \Phi}{\Delta x}\right)=\frac{k_{r\,i+1/2}\left(\Phi_{i+1}-\Phi_i\right)-k_{r\,i-1/2}\left(\Phi_i-\Phi_{i-1}\right)}{\Delta x^2}$$

$$\frac{\Delta_x}{\Delta x}\left(k_r\frac{\Delta_x P_c}{\Delta x}\right)=\frac{k_{r\,i+1/2}\left(P_{c,i+1}-P_{c,i}\right)-k_{r\,i-1/2}\left(P_{c,i}-P_{c,i-1}\right)}{\Delta x^2}$$

$$\frac{\Delta_t S}{\Delta t}=\frac{S_i^{n+1}-S_i^n}{\Delta t}$$

The IMPES stencil is as follows. The potential is calculated at the *n+1* th time level while using the relative permeabilities and capillary pressures at the *n* th time level. Next the saturations are updated using the new potentials but

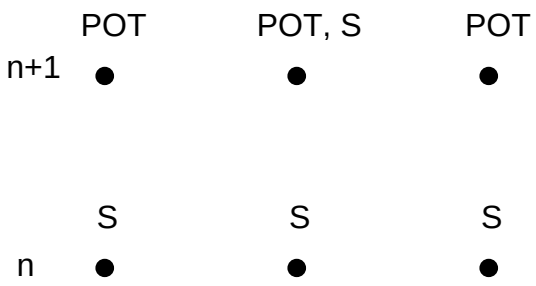

Fig. 4.1 IMPES finite difference stencil

relative permeabilities and capillary pressures at the *n* th time level. The resulting finite difference equation is as follows.

$$\lambda_{i+1/2}^{n}\left(\Phi_{i+1}^{n+1}-\Phi_{i}^{n+1}\right)-\lambda_{i-1/2}^{n}\left(\Phi_{i}^{n+1}-\Phi_{i-1}^{n+1}\right)=\frac{1}{M}\left[k_{rd,i+1/2}^{n}\left(P_{c,i+1}^{n}-P_{c,i}^{n}\right)-k_{rd,i-1/2}^{n}\left(P_{c,i}^{n}-P_{c,i-1}^{n}\right)\right]$$

$$S_i^{n+1}-S_i^{n}=\frac{\Delta t}{\Delta x^2}\left[k_{ri\ i+1/2}^{\ n}\left(\Phi_{i+1}^{n+1}-\Phi_{i}^{n+1}\right)-k_{ri\ i-1/2}^{\ n}\left(\Phi_{i}^{n+1}-\Phi_{i-1}^{n+1}\right)\right]$$

Initial and Boundary Conditions

Initial Condition: $S_i^0 = \dfrac{S_i^{IC} - S_{ir}}{1 - S_{dr} - S_{ir}}$.

x=0 BC: $k_{rd}{}^{n}_{1/2} = 0, \quad \lambda^{n}_{1/2} = 1$

$\lambda^{n}_{1/2}\left(\Phi_1^{n+1} - \Phi_0^{n+1}\right) = -\Delta x, \qquad k_{ri}{}^{n}_{1/2}\left(\Phi_1^{n+1} - \Phi_0^{n+1}\right) = -\Delta x$

x=1 BC:

$\Phi^{n+1}_{NX+1/2} = 0,$

for $i = NX$, replace $\dfrac{\Phi^{n+1}_{i+1} - \Phi^{n+1}_{i}}{\Delta x}$ with $\dfrac{\Phi^{n+1}_{NX+1/2} - \Phi^{n+1}_{i}}{\Delta x/2} = \dfrac{2\left(\Phi^{n+1}_{i+1} - \Phi^{n+1}_{i}\right)}{\Delta x}$

and $\Phi^{n+1}_{NX+1} = 0$

also, same for $P_C(S)$ at *i=NX*.

Prevent reverse flow of invading phase by specifying $k_{ri}{}^{n}_{NX+1/2} = 0, \quad \text{if} \quad \Phi^{n}_{NX} < 0$

Matrix Inversion

The procedure is implicit in the potential and a system of equations must be solved. A direct LU inversion can be made on the tridiagonal matrix.

$$\begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \cdots & \cdots & \cdots & \\ & & a_{NX-1} & b_{NX-1} & c_{NX-1} \\ & & & a_{NX} & b_{NX} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \cdots \\ \Phi_{NX-1} \\ \Phi_{NX} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \cdots \\ r_{NX-1} \\ r_{NX} \end{bmatrix}$$

Stability of IMPES Procedure

We derived earlier the stability condition for explicit calculation of the Buckley-Leverett equation. Here we will examine the additional stability condition imposed by including capillary pressure with the IMPES formulation. The dimensionless differential equations are as follows.

$$\frac{\partial}{\partial x}\left[\left(k_{ri}+\frac{k_{rd}}{M}\right)\frac{\partial \Phi}{\partial x}-\frac{k_{rd}}{M}\frac{\partial P_c}{\partial x}\right]=0$$

$$\frac{\partial S_i}{\partial t}-\frac{\partial}{\partial x}\left[k_{ri}(S)\frac{\partial \Phi}{\partial x}\right]=0$$

Since the divergence in the first equation is zero, the quantity in the bracket must be independent of x and the boundary condition requires that it be equal to unity. The potential gradient can be determined from this equation and the expression can be substituted into the second equation to eliminate the potential gradient.

$$\frac{\partial S}{\partial t}+\frac{\partial}{\partial x}\left[\frac{k_{ri}}{k_{ri}+k_{rd}/M}\left(1-\frac{k_{rd}}{M}\frac{\partial P_c}{\partial x}\right)\right]=0$$

You may recognize that in the absence of the capillary pressure, this equation is the Buckley-Leverett equation. This equation may have a different sign for the capillary pressure term compared to some other writers but remember that we are defining the capillary pressure to be negative for imbibition processes and positive for drainage processes. We are currently interested only in the stability due to the explicit evaluation of the capillary pressure. Retaining only the term with the second spatial derivative gives the following.

$$\frac{\partial S}{\partial t}\approx\frac{k_{ri}\,k_{rd}}{M\,k_{ri}+k_{rd}}\frac{dP_c}{dS}\frac{\partial^2 S}{\partial x^2}$$

This equation can be expressed in finite difference as follows.

$$S_i^{n+1}=\beta S_{i-1}^n+(1-2\beta)S_i^n+\beta S_{i+1}^n$$

where

$$\beta=\frac{\Delta t}{\Delta x^2}\frac{k_{ri}\,k_{rd}}{M\,k_{ri}+k_{rd}}\frac{dP_c}{dS}=\frac{\Delta t}{\Delta x^2}\frac{k_{ri}\,k_{rd}}{M\,k_{ri}+k_{rd}}N_{P_c}\frac{dJ}{dS}$$

We can determine the stability condition by applying the Gerschgorin theorem.

$$\frac{\Delta t}{\Delta x^2}<\frac{\left(M\,k_{ri}+k_{rd}\right)}{2k_{ri}\,k_{rd}\,N_{P_c}\dfrac{dJ}{dS}}$$

Assignment 4.1 IMPES formulation of Buckley Leverett problem
Solve the problem of Assignment 1.1 using the IMPES formulation. Verify your code by comparing the fractional flow and recovery with the fractional flow formulation for NX=10. Verify that material balance is exactly satisfied. Include your code with the assignment.

The tridiagonal inversion and capillary pressure routines are in the file, *impes.sub* on owlnet userid *gjh*, directory *class*.

Assignment 4.2 Stability of IMPES formulation with drainage capillary pressure
Do the problem of Assignment 4.1 but with injection of oil into 100% water saturated system, $N_{P_c} = 1.0$, $C_1 = +0.17$, $C_2 = 0.29$ (i.e., Berea drainage). Let NX = 10, 20, 40, 80. First try with $\frac{\Delta t}{\Delta x} = 0.1$; what happens? Next try with $\frac{\Delta t}{\Delta x^2} = 0.5$. Why the difference? Plot profile at t=2.0 and compare with MOC solution. Plot fractional flow recovery efficiency and compare with MOC up to t=5.

Assignment 4.3 Imbibition capillary pressure
Do the problem of Assignment 4.1 but with $N_{P_c} = 1.0$, $C_1 = -0.17$, $C_2 = 0.29$ and S^{IC}=0.2. Let NX = 10, 20, 40, 80 and $\frac{\Delta t}{\Delta x^2} = 0.5$. Plot profile at t=0.5. Plot fractional flow and recovery efficiency to t=2. Define recover efficiency as fraction of movable pore volume in this case.

Assignment 4.4 Counter-current imbibition
Modify code to allow flow reversal of the invading phase, i.e. allow inflow of the invading fluid from the outflow end and determine upstream direction for the invading phase relative permeability. Let: M=2.0, n=1.5, N_{P_c} =100., C1=-0.17, C2=0.29, S^{IC}=0.2, NX=20, $\frac{\Delta t}{\Delta x^2} = 0.005$. Plot profiles every 0.005 to t=0.04. Plot recovery efficiency to t=0.04. Plot recovery efficiency as function of the square root of ($N_{P_c} \times$time). What is the reason for the two linear regions?

Reading Assignment Chapters 5 and 6 in Reservoir Simulation

Simulation of Centrifuge Displacement with Semi-Implicit Formulation

Centrifuge displacement is usually conducted under conditions where the capillary pressure is significant and the sample length is short. The stability limitation of the IMPES formulation may be very severe for centrifuge displacement experiments. Thus the equations will be formulated with a semi-implicit formulation. To be fully implicit, a nonlinear system of equations will need to be solved for the dependent variables at the new time level. The semi-implicit formulation approximates the nonlinear terms by the value at the old time level plus the differential of the term multiplied by the change of the dependent variable over the time step.

$$k_r^{n+1} \approx k_r^n + \frac{dk_r}{dS}\left(S^{n+1} - S^n\right)$$
$$= k_r^n + \frac{dk_r}{dS} DS$$

$$P_c^{n+1} = P_c^n + \frac{dP_c}{dS}\left(S^{n+1} - S^n\right)$$
$$= P_c^n + \frac{dP_c}{dS} DS$$

The saturation change over the time step, *DS*, is a dependent variable that is solved simultaneously with the pressure. In the one dimensional case, the coefficient matrix is block tridiagonal.

In centrifuge displacement, the rock sample is immersed in the invading phase and the sample is supported such that the zero capillary pressure boundary condition at the outflow end is satisfied. Instead of a gravitational field, the buoyancy is due to a centrifugal field that is a function of both time and position. It is a function of time because it takes a finite amount of time for the centrifuge to reach the set point rotational speed. Also, the speed is changed in multi-speed experiments. It is a function of position because different parts of the sample will be at a different distance from the axis of the centrifuge.

$$u_d = -\frac{k\, k_{rd}(S)}{\mu_d}\left[\frac{\partial p_d}{\partial x} - \rho_d\, \omega^2(t) R(x)\right]$$

$$u_i = -\frac{k\, k_{ri}(S)}{\mu_i}\left[\frac{\partial p_i}{\partial x} - \rho_i\, \omega^2(t) R(x)\right]$$

$$\phi\frac{\partial S_d}{\partial t} + \frac{\partial u_d}{\partial x} = 0$$

$$\phi\frac{\partial S_i}{\partial t} + \frac{\partial u_i}{\partial x} = 0$$

$$S_d + S_i = 0$$

$$p_i - p_d = P_c(S)$$

$$S_d(x,0) = S_{do}$$

$$u_d(0,t) = 0, \quad \text{inflow end}$$

$$P_c(L,t) = 0, \quad \text{outflow end}$$

$$p_i(L,t) - p_i(0,t) = \frac{1}{2}\rho_i\, \omega^2(t)\left[R^2(L) - R^2(0)\right]$$

The initial condition does not have to be uniform but will be given as a uniform value here. However, the invading phase does not have to be immobile at the initial condition. In the case of imbibition displacement, the initial condition is determined by the capillary pressure and average saturation attained during the prior drainage experiment. The condition of zero flux of the displaced phase at the inflow end assumes that outflow of the displaced phase due to counter-current imbibition will not occur because this would result in a layer of displaced phase shielding the inflow phase from the invading phase. The outflow boundary condition of zero capillary pressure assumes that the displaced phase does not wet the end piece of the core holder. No restriction is placed on inflow of the invading phase from the outflow end. This can occur during imbibition experiments. The pressure boundary condition assumes that the invading phase is in hydrostatic equilibrium outside the core.

A natural change of variables is to replace the phase pressures with the flow potential of the invading phase and the capillary pressure.

$$\Phi(x,t) = p_i(x,t) - p_i(0,t) - \frac{1}{2}\rho_i\, \omega^2(t)\left[R^2(x) - R^2(0)\right]$$

$$\Phi(0,t) = \Phi(L,t) = 0$$

The flow potential has a convenient boundary condition of zero at both ends, and it is needed to determine the upstream direction for the invading phase. The variables are made dimensionless or normalized as follows:

$$S = \frac{S_d - S_{dr}}{S_{do} - S_{dr}}$$

$$k_{rdD} = \frac{k_{rd}}{k_{rd}^o}$$

$$k_{riD} = \frac{k_{ri}}{k_{ri}^o}$$

$$M = \frac{k_{ri}^o}{k_{rd}^o}\frac{\mu_d}{\mu_i}$$

$$\omega_D(t) = \frac{\omega(t)}{\omega_o}$$

$$R_D(x) = \frac{R(x)}{R_o}$$

$$x_D = \frac{x}{L}$$

$$t_D = \frac{k\, k_{rd}^o\, \Delta\rho\, \omega_o^2 R_o}{\phi\left(S_{do} - S_{dr}\right) L \mu_d} t$$

$$P_{cD} = \frac{P_c}{\Delta\rho\, \omega_o^2 R_o L} = \frac{\sigma\, J(S)}{\Delta\rho\, \omega_o^2 R_o L\sqrt{\frac{k}{\phi}}}$$

$$\Phi_D = \frac{\Phi}{\Delta\rho\, \omega_o^2 R_o L}$$

After dropping the subscript D, the dimensionless equations expressing the conservation of mass and momentum during the displacement are as follows.

$$\frac{\partial}{\partial x}\left[\left(k_{rd} + M\, k_{ri}\right)\frac{\partial \Phi}{\partial x} - k_{rd}\left(\omega^2(t) R(x) + \frac{\partial P_c}{\partial x}\right)\right] = 0$$

$$\frac{\partial}{\partial x}\left[k_{rd}\left(\frac{\partial \Phi}{\partial x} - \frac{\partial P_c}{\partial x} - \omega^2(t) R(x)\right)\right] - \frac{\partial S}{\partial t} = 0$$

The boundary conditions are homogeneous and thus do not change on being made dimensionless. The saturation initial condition is equal to unity. In the following derivations, the product $\omega^2(t) R(x)$ is assumed to be unity only for clarity of the equations. It is easy to include any specified function for this product to represent a changing centrifugal field.

Finite Difference Approximation

The differential equations are approximated with a finite difference approximation that uses upstream weighting of the mobility, is implicit in the potential and semi-implicit in the saturation. A semi-implicit rather than a fully implicit, Newtonian iteration method is used. The semi-implicit method is one iteration of the Newtonian iteration formulation. Since the accumulation term is linear in the dependent variable, (i.e., saturation) iterations are not required for the method to conserve matter. It is not desirable to be fully implicit in the in the mobility terms when it is not needed for stability because the implicit in saturation formulation has the spatial and temporal discretization errors being additive, while the explicit formulation has the errors partially canceling. Thus only a fraction of the implicit contribution to mobility is used as needed for stability. It is necessary to solve the potential and saturation simultaneously for the case of large mobility ratio ($M=10^6$) and drainage capillary pressure curves with a narrow pore size distribution (a=0.1 in the Thomeer model). This is accomplished by solving the system of equations with a block tridiagonal algorithm.

The finite difference equations (before being modified for boundary conditions) are as follows.

$$
\begin{aligned}
&\lambda_{i-1/2}\Phi_{i-1} - \left(\lambda_{i-1/2} + \lambda_{i+1/2}\right)\Phi_i + \lambda_{i+1/2}\Phi_{i+1} \\
&-k'_{rd,i-1/2}\left[\left(\Phi^*_i - \Phi^*_{i-1}\right) - \Delta x - \left(P_{c,i} - P_{c,i-1}\right)\right] DS_{i-1/2,d} \\
&+k'_{rd,i+1/2}\left[\left(\Phi^*_{i+1} - \Phi^*_i\right) - \Delta x - \left(P_{c,i+1} - P_{c,i}\right)\right] DS_{i+1/2,d} \\
&-M\,k'_{ri,i-1/2}\left(\Phi^*_i - \Phi^*_{i-1}\right) DS_{i-1/2,i} + M\,k'_{ri,i+1/2}\left(\Phi^*_{i+1} - \Phi^*_i\right) DS_{i+1/2,i} \\
&-k_{rd,i-1/2}\,P_c{}'_{,i-1}\,DS_{i-1} + \left(k_{rd,i-1/2} + k_{rd,i+1/2}\right) P_c{}'_{,i}\,DS_i - k_{rd,i+1/2}\,P_c{}'_{,i+1}\,DS_{i+1} \\
&= \Delta x\left\{k_{rd,i+1/2}\left[1 + \left(P_{c,i+1} - P_{c,i}\right)/\Delta x\right] - k_{rd,i-1/2}\left[1 + \left(P_{c,i} - P_{c,i-1}\right)/\Delta x\right]\right\}
\end{aligned}
$$

$$f^{*}\Delta x k'_{rd,i-1/2}\left[1+\frac{\left(P_{c,i}-P_{c,i-1}\right)}{\Delta x}-\frac{\left(\Phi^{*}_{i}-\Phi^{*}_{i-1}\right)}{\Delta x}\right]DS_{i-1/2,d}$$

$$-f^{*}\Delta x k'_{rd,i+1/2}\left[1+\frac{\left(P_{c,i+1}-P_{c,i}\right)}{\Delta x}-\frac{\left(\Phi^{*}_{i+1}-\Phi^{*}_{i}\right)}{\Delta x}\right]DS_{i+1/2,d}$$

$$-k_{rd,i-1/2}P'_{c,i-1}DS_{i-1}+\left[-\frac{\Delta x^{2}}{\Delta t}+\left(k_{rd,i-1/2}+k_{rd,i+1/2}\right)P'_{c,i}\right]DS_{i}-k_{rd,i+1/2}P'_{c,i+1}DS_{i+1}$$

$$+k_{rd,i-1/2}\Phi_{i-1}-\left(k_{rd,i-1/2}+k_{rd,i+1/2}\right)\Phi_{i}+k_{rd,i+1/2}\Phi_{i+1}$$

$$=\Delta x\left\{k_{rd,i+1/2}\left[1+\frac{\left(P_{c,i+1}-P_{c,i}\right)}{\Delta x}\right]-k_{rd,i-1/2}\left[1+\frac{\left(P_{c,i}-P_{c,i-1}\right)}{\Delta x}\right]\right\}$$

$$0.3\frac{\Delta t}{\Delta x}\le f^{*}\le 0.5$$

The quantity, *DS* is the saturation change over the time step. It is subscript at a fractional grid location for the implicit contribution to the relative permeabilities. The appropriate grid location is the upstream location corresponding to the location where the relative permeability is evaluated. The upstream location must be determined for the invading and displaced phases by examining the sign of the respective potential differences between the grid blocks. (With the boundary conditions stated here the upstream direction for the displaced phase is from the inflow to the outflow end. However, this may not be the case for a different set of boundary conditions.) When flow reversals are occurring, the time level at which the direction of the potential gradient is determined may be important. One approach is to iterate twice on the potential equation (tridiagonal matrix) without including the implicit saturation contributions to determine the upstream direction for each phase. The current upstream relative permeability is used in each iteration. The entire system is then solved by inverting the block tridiagonal matrix. The potential, Φ^{*} for the implicit saturation terms is the potential determined after these two iterations.