Chapter 1 Finite Difference for Fractional Flow Equation

Reading assignment: Reservoir Simulation (Mattax and Dalton 1990), Chapter 1, 2, 5, and Appendix B.

Differential Equation

The one dimensional, two phase Buckley-Leverett displacement will be solved using the finite difference approximation. The Buckley-Leverett solution will be used to evaluate the approximation errors of the finite difference method.

$$\frac{\partial S_j}{\partial t} = -\frac{u_T}{\phi}\frac{\partial f_j}{\partial x}, \qquad x > 0, \quad t > 0, \quad j = w, o$$

$$S_j(x,0) = \text{I.C.}$$

$$f_j(0,t) = \text{B.C.}$$

Let: $S = \dfrac{S_w - S_{wr}}{1 - S_{or} - S_{wr}}$

$$x_D = \frac{x}{L}$$

$$t_D = \frac{u_T}{(1 - S_{or} - S_{wr})\,\phi\, L}\; t$$

$$= \frac{Q}{(1 - S_{or} - S_{wr})\; V_p}$$

Drop subscripts:

$$\frac{\partial S}{\partial t} = -\frac{\partial f}{\partial x}$$

This equation could be expanded as

$$\frac{\partial S}{\partial t} = -\frac{df}{dS}\frac{\partial S}{\partial x}$$

but the finite difference approximation to this latter equation will not be conservative. We will show later that in order for the equations to be conservative, the flux leaving one grid block must equal the flux entering the adjacent grid block.

Finite Difference Approximations

The time derivative is approximated with the following finite difference expressions.

$$\frac{\partial S}{\partial t} \approx \frac{S^{n+1} - S^n}{\Delta t} = \frac{\Delta_t S}{\Delta t}$$

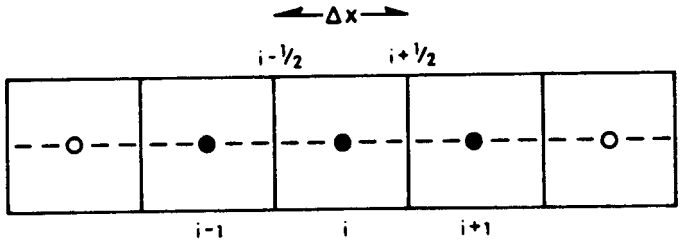

Fig. 1.1 Block centered grid

Several different approximations can be used for the spatial derivative. Figure 1.1 illustrates the "block centered" grid that is commonly used in reservoir simulation rather than the "point centered" grid commonly used in class room instruction. The dependent variable is defined at the center of the grid block and the fluxes are evaluated across the faces separating the grid blocks. The divergence of the flux in grid block *i* expressed in finite difference is the net efflux from the grid block. The illustrations here are for equal grid spacing. The concepts are the same for unequal grid spacing but the equations have more detail than needed here.

$$\left(\frac{\partial f}{\partial x}\right)_i \approx \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x}$$

Central difference:

But *f* is a function of *S* which is evaluated at x_{i-1}, x_i, x_{i+1}, ..., thus the central difference approximation approximates the value at *i+1/2* and *i-1/2* with an average of the value on either side.

$$\left(\frac{\partial f}{\partial x}\right)_i \approx \frac{\frac{1}{2}\left(f_{i+1} + f_i\right) - \frac{1}{2}\left(f_i + f_{i-1}\right)}{\Delta x}$$

$$= \frac{f_{i+1} - f_{i-1}}{2\Delta x}$$

Even though the central difference approximation is a second order correct method, it suffers from problems such as sharpening a front more than it should and even causing over-shoot behind a shock front.

1 Point Upstream weighting:

The one point upstream weighting scheme approximates the value of *f* at *i+1/2* and *i-1/2* with the value in the grid block on the upstream side, e.g. from a smaller value of *x* in this case.

$f_{i+1/2} \approx f_i$, *and* $f_{i-1/2} \approx f_{i-1}$ If flow is left to right.

$$\left(\frac{\partial f}{\partial x}\right)_i \approx \frac{f_i - f_{i-1}}{\Delta x}$$

This approximation is first order correct, i.e., O(Δx). The truncation error from this approximation will result in "numerical dispersion" but it has none of the over-shoot problems of the central difference approximation.

2-Point Upstream Weighting

Rather than using the upstream value to approximate $f_{i-1/2}$, the 2-point upstream method uses the two points upstream of *i-1/2* to approximate the value of the flux at *i-1/2*.

$$\left(\frac{\partial f}{\partial x}\right)_i \approx \frac{3f_i - 4f_{i-1} + f_{i-2}}{2\Delta x}$$ (note: do not use this expression directly in code)

Certain precautions apply to this method (Todd, O'Dell, Hirasaki 1972).

Saturation Profiles with Different Weighting Methods

The saturation profiles from the central difference or "mid point weighting" and the upstream weighting are compared with the Buckley-Leverett profile for a mobility ratio of 0.455 in Fig. 1.2. The saturation distribution of water along the system at 0.32 PV of water injection is shown. The solid curve is the Buckley-Leverett solution and the symbols are the finite difference results. It is readily apparent that the use of mid point weighting is completely unacceptable for this case. Aside from the overshoot in the predicted saturation profile, the predicted location of the front is grossly in error. The single point and two point weighting are the upstream weighting methods mentioned above.

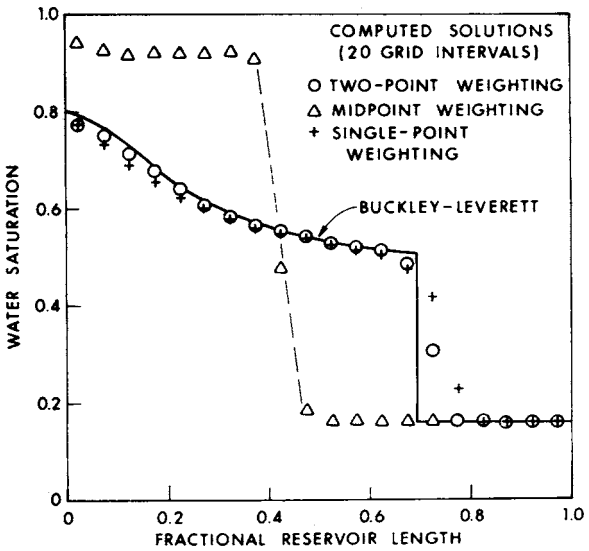

Fig. 1.2 Comparison of mobility weighting schemes (Todd, O'Dell, Hirasaki 1972)

Explicit or Implicit Formulation

The accuracy of the finite difference approximation for the time derivative and the stability of the procedure is dependent on the time level at which the flux terms in the conservation equation is evaluated. Three common time levels for evaluating the flux terms are shown on Figure 1.3. The points in space where the dependent variables are evaluated are denoted by the index i for this one-dimensional illustration. It has been assumed that the spatial difference terms in the flux require the values at x_{i-1}, x_i, and x_{i+1} for the finite difference equation at x_i. The points in space and time at which the values of the dependent variable appear in the finite difference equation are shown as the solid disks and are connected by the solid lines.

The "**forward difference**" formulation evaluates the spatial differences for the flux terms at the old time level, t_n using the known values of the dependent variable. Thus, the difference approximation in time moves the solution forward from t_n to t_{n+1}. Since each finite difference equation has only one unknown (the value of the dependent variable at x_i and t_{n+1}) the formulation is said to be an "**explicit**" procedure.

The "**backward difference**" formulation evaluates the spatial differences for the flux terms on the new time level, t_{n+1}, using the as yet unknown values of the dependent variable. Thus, the difference approximation in time couples the finite difference equation backwards in time to the known value of the dependent variable at the old time level, t_n. Each finite difference equation has several (three in this illustration) unknown values of the dependent variable at the new time level. Since the dependent variable at the new time level has to be computed by solving a system of equations, this formulation is said to be an "**implicit**" procedure.

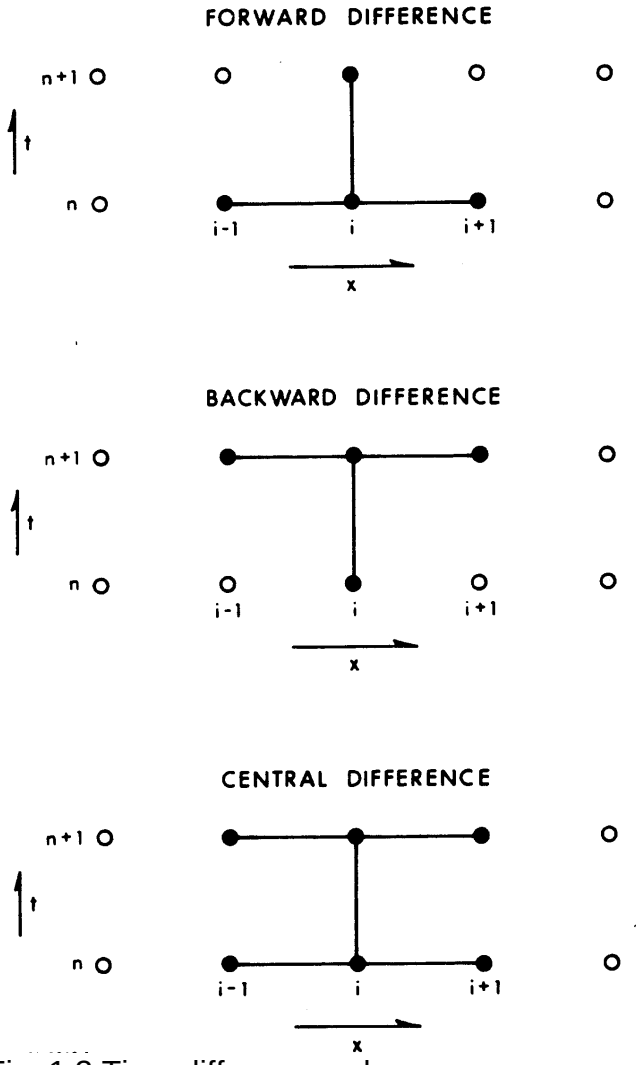

Fig. 1.3 Time difference schemes

The "**central difference**" approximation evaluates the flux terms at both the old and new time levels and uses an average value. It is central in the sense that the finite difference equation is centered between the old and new time levels. This method is also often known as the **Crank-Nicholson** procedure. It is also implicit as each finite difference equation will contain several unknown dependent variables. This method has a time truncation error that is second order correct. Stability analysis for this method applied to the parabolic problems show it to be unconditionally stable but is conditionally oscillatory if the problem has derivative (Neumann) boundary conditions.

A combination of forward difference, backward difference, and/or central difference procedures may be used to solve the finite difference equations. For example, the implicit pressure-explicit saturation (**IMPES**) procedure uses the backward difference (implicit) procedure for pressure and the forward difference (explicit) procedure for saturations. On the other hand, the totality implicit simulator such as for well coning uses the backward difference (implicit) procedure for both pressure and saturations.

Time Truncation Error

The accuracy of the different procedures for approximating the time derivative can be estimated by evaluating the "truncation error" arising from approximating the derivative with a finite difference. The term "truncation error" is sometimes defined as the **global** error between the solution of difference equation and the differential equation. However, the error will be defined here as the **local** error between the derivative and the finite difference approximation as determined from a truncated Taylor's series.

The truncation error of the **forward difference** (FD) can be approximated by a truncated Taylor's series expansion about t_n. As an example let the saturation be the dependent variable.

$$S_{n+1} = S_n + \Delta t \left(\frac{\partial S}{\partial t} \right)_n + \frac{\Delta t^2}{2} \frac{\partial^2 S}{\partial t^2}(\bar{t})$$

where

$$t_n \le \bar{t} \le t_{n+1}$$

Solving for the difference approximation, we have

$$\frac{S_{n+1} - S_n}{\Delta t} = \left(\frac{\partial S}{\partial t} \right)_n + \frac{\Delta t}{2} \frac{\partial^2 S}{\partial t^2}(\bar{t})$$

Thus the error in approximating the derivative by a finite difference is

$$e_{FD} = \frac{\Delta t}{2}\frac{\partial^2 S}{\partial t^2}(\bar{t})$$

This term goes to zero in proportion to Δt, so the error of this approximation is denoted as "of the first order in Δt" or "of order Δt."

$$e_{FD} = O(\Delta t)$$

By using a truncated Taylor's series about t_{n+1} and $1/2(t_n + t_{n+1})$, the error of the approximation with backward difference (BD) and central difference (CD) can be estimated.

Backward Difference, BD

$$\frac{S_{n+1} - S_n}{\Delta t} = \left(\frac{\partial S}{\partial t}\right)_{n+1} - \frac{\Delta t}{2}\frac{\partial^2 S}{\partial t^2}(\bar{t})$$

$$e_{BD} = -\frac{\Delta t}{2}\frac{\partial^2 S}{\partial t^2}(\bar{t})$$

$$= O(\Delta t)$$

Central Difference, CD

$$\frac{S_{n+1} - S_n}{\Delta t} = \left(\frac{\partial S}{\partial t}\right)_{n+1/2} + \frac{\Delta t^2}{48}\left(\frac{\partial^3 S}{\partial t^3}(\bar{t}) + \frac{\partial^3 S}{\partial t^3}(\bar{t}^*)\right)$$

$$e_{CD} = \frac{\Delta t^2}{48}\left(\frac{\partial^3 S}{\partial t^3}(\bar{t}) + \frac{\partial^3 S}{\partial t^3}(\bar{t}^*)\right)$$

$$= O(\Delta t^2)$$

The above equations imply that the truncation error of the forward and backward differences goes to zero in proportion to Δt and the truncation error of the central difference goes to zero in proportion to the square of Δt. The magnitude of the truncation error for a given value of Δt is a function of the second or third derivative of the dependent variable with time.

Although the central difference approximation appears to be the most accurate as Δt becomes small, it is necessary to also consider stability and complexity of coding the procedure when deciding on a procedure for a particular application. The forward difference is the simplest and the central difference is the most complex method to code. However, the forward difference procedure has a stability limitation that limits the time step size and the central difference procedure has oscillations that may limit the time step size.

Most standard "black oil" simulators are explicit in saturation and implicit in pressure (i.e., IMPES). The severe time step size limitation associated with single well coning problems requires a formulation that is implicit in both the saturations and pressure (i.e., fully or semi-implicit).

Spatial Truncation Errors

The truncation error for the **one point upstream weighting** is derived **assuming the upstream direction is always towards lower values of *x***.

$$f_{i-1} = f_i - \Delta x\left(\frac{\partial f}{\partial x}\right)_i + \frac{\Delta x^2}{2}\frac{\partial^2 f}{\partial x^2}(\bar{x})$$

$$\frac{f_i - f_{i-1}}{\Delta x} = \left(\frac{\partial f}{\partial x}\right)_i - \frac{\Delta x}{2}\frac{\partial^2 f}{\partial x^2}(\bar{x})$$

$$= \left(\frac{\partial f}{\partial x}\right)_i + O(\Delta x)$$

The **two point upstream weighting** truncation error is derived assuming that the upstream direction is toward lower values of *x*.

$$f_{i-1} = f_i - \Delta x\left(\frac{\partial f}{\partial x}\right)_i + \frac{\Delta x^2}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_i - \frac{\Delta x^3}{3!}\left(\frac{\partial^3 f}{\partial x^3}\right)(\bar{x})$$

$$f_{i-2} = f_i - 2\Delta x\left(\frac{\partial f}{\partial x}\right)_i + \frac{4\Delta x^2}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_i - \frac{8\Delta x^3}{3!}\left(\frac{\partial^3 f}{\partial x^3}\right)(\bar{x}^*)$$

Eliminate the second derivative term by multiplying the first equation by 4 and subtracting from the second equation.

$$f_{i-2} - 4f_{i-1} = -3f_i + 2\Delta x\left(\frac{\partial f}{\partial x}\right)_i + O(\Delta x^3)$$

$$\left(\frac{\partial f}{\partial x}\right)_i = \frac{3f_i - 4f_{i-1} + f_{i-2}}{2\Delta x} + O(\Delta x^2)$$ (note: do not use this expression directly in code)

This method is called 2 point upstream weighting because it uses two upstream points for an approximation to $f_{i-1/2}$ and $f_{i+1/2}$.

$$\frac{f_{i+1/2} - f_{i-1/2}}{\Delta x} = \left(\frac{\partial f}{\partial x}\right)_i + O\left(\Delta x^2\right)$$

$$f_{i-1/2} = \frac{3}{2} f_{i-1} - \frac{1}{2} f_{i-2} + O\left(\Delta x^2\right)$$

$$f_{i+1/2} = \frac{3}{2} f_i - \frac{1}{2} f_{i-1} + O\left(\Delta x^2\right)$$

When implementing the 2-point upstream method, it is necessary to constrain the fractional flow such that it is positive and less than unity.

The **central difference approximation** is second order accuracy but has the overshoot problem mentioned earlier.

$$f_{i+1} = f_i + \Delta x\left(\frac{\partial f}{\partial x}\right)_i + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_i + \frac{(\Delta x)^3}{6}\frac{\partial^3 f}{\partial x^3}(\bar{x})$$

$$f_{i-1} = f_i - \Delta x\left(\frac{\partial f}{\partial x}\right)_i + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_i - \frac{(\Delta x)^3}{6}\frac{\partial^3 f}{\partial x^3}(\bar{x}*)$$

$$f_{i+1} - f_{i-1} = 2\Delta x\left(\frac{\partial f}{\partial x}\right)_i + \frac{(\Delta x)^3}{6}\left[\frac{\partial^3 f}{\partial x^3}(\bar{x}) + \frac{\partial^3 f}{\partial x^3}(\bar{x}*)\right]$$

$$\frac{f_{i+1} - f_{i-1}}{2\Delta x} = \left(\frac{\partial f}{\partial x}\right)_i + \frac{(\Delta x)^2}{12}\left[\frac{\partial^3 f}{\partial x^3}(\bar{x}) + \frac{\partial^3 f}{\partial x^3}(\bar{x}*)\right]$$

Numerical Dispersion

We will examine the effect of the truncation error of the explicit and implicit one point upstream weighting formulation by examining the expression for the truncation error of the entire finite difference equation. We will first examine the **explicit** formulation with **one point upstream** weighting. Substitute the finite difference expressions for the time and spatial differentials into the differential equation. Subscripts will be used for the spatial index and superscripts will be used for the time index.

$$\frac{\left(S_i^{n+1} - S_i^n\right)}{\Delta t} + \frac{\left(f_i{}^n - f_{i-1}^n\right)}{\Delta x} =$$

$$\left(\frac{\partial S}{\partial t}\right)_i^n + \left(\frac{\partial f}{\partial x}\right)_i^n + \frac{\Delta t}{2}\frac{\partial^2 S}{\partial t^2}(\bar{t}) - \frac{\Delta x}{2}\frac{\partial^2 f}{\partial x^2}(\bar{x})$$

We will simplify the analysis by assuming that the fractional flow expression is a linear function of saturation, i.e.

$$\frac{df}{dS} = f' = constant$$

The differential equation is now expressed in terms of derivatives of S only.

$$\frac{\partial S}{\partial t} + f'\frac{\partial S}{\partial x} = 0$$

$$\frac{\left(S_i^{n+1} - S_i^n\right)}{\Delta t} + f'\frac{\left(S_i^n - S_{i-1}^n\right)}{\Delta x} =$$

$$\left(\frac{\partial S}{\partial t}\right)_i^n + f'\left(\frac{\partial S}{\partial x}\right)_i^n + \frac{\Delta t}{2}\frac{\partial^2 S}{\partial t^2}(\bar{t}) - \frac{\Delta x}{2}f'\frac{\partial^2 S}{\partial x^2}(\bar{x})$$

$$\frac{\partial^2 S}{\partial t^2} = -f'\frac{\partial^2 S}{\partial t \partial x}$$

$$= \left(f'\right)^2\frac{\partial^2 S}{\partial x^2}$$

Substituting into the previous equation we have,

$$\frac{\left(S_i^{n+1}-S_i^n\right)}{\Delta t}+f'\frac{\left(S_i^n-S_{i-1}^n\right)}{\Delta x}$$

$$=\left(\frac{\partial S}{\partial t}\right)_i^n+f'\left(\frac{\partial S}{\partial x}\right)_i^n+\frac{\Delta t}{2}\left(f'\right)^2\frac{\partial^2 S}{\partial x^2}-\frac{\Delta x}{2}f'\frac{\partial^2 S}{\partial x^2}$$

$$=\left(\frac{\partial S}{\partial t}\right)_i^n+f'\left(\frac{\partial S}{\partial x}\right)_i^n+f'\left(\frac{f'\Delta t}{2}-\frac{\Delta x}{2}\right)\frac{\partial^2 S}{\partial x^2}$$

$$=\left(\frac{\partial S}{\partial t}\right)_i^n+f'\left(\frac{\partial S}{\partial x}\right)_i^n-D\frac{\partial^2 S}{\partial x^2}$$

Where the coefficient *D* is

$$D=\frac{f'}{2}\left(\Delta x-f'\Delta t\right)$$

We now see that the finite difference equation is equivalent to the **convective-diffusion** equation with the diffusion coefficient a function of Δx and Δt. We know that the solution of this equation with *D=0* and constant *f'* is an indifferent, step wave. The effect of the truncation error is to cause an apparent dispersion of the displacement front. With the explicit formulation the spatial and time truncation errors tend to cancel each other since they have opposite signs. However, if $\frac{\Delta t}{\Delta x}>\frac{1}{f'}$, then the value of *D* will be negative. The solution to the diffusion equation with negative diffusion equation is unstable. We also will see that a stability analysis of the finite difference equation shows that the scheme is unstable if *D* is negative.

An equivalent analysis of the **implicit** formulation with **one point upstream** weighting show the coefficient to be of the following form.

$$D=\frac{f'}{2}\left(\Delta x+f'\Delta t\right)$$

The spatial and time truncation errors are additive for the implicit formulation. Thus the coefficient will never become negative. Also the implicit formulation is unconditionally stable.

The methods with second order spatial approximations have truncations errors that can not be represented as a dispersion term (i.e., coefficient of second spatial derivative). The second order spatial approximation methods have the truncation error term that corresponds to the third order spatial derivative in the Taylor series expansion.

Recovery Efficiency

The oil recovery efficiency can be calculated in two ways, either as a change in the average saturation in the system or as a cumulative production. The former is calculated by a numerical integration in space and the latter by a numerical integration in time. If the method is conservative (and no mistake in coding), the two methods for calculating the recovery efficiency will agree within round-off errors.

Assignment 1-1 Finite difference solution to the Buckley Leverett equation.

$$k_{r1} = k_{r1}^{o} S^{n_1}$$
$$k_{r2} = k_{r2}^{o}(1-S)^{n_2}$$

$$n1 = n2 = 1.5$$
$$\alpha = 0$$
$$M = 2.0$$
$$S(x,0) = 0$$
$$f(0,t) = 1.0$$

Solve for (1) saturation profile at *t=0.5*, (2) effluent history of *f*, (3) recovery efficiency up to *t=2.0*. Compare with the method of characteristics (MOC) solution. Let *NX= 1.0/∆x=100* and *∆t/∆x=0.1*.

If you do not have the method of characteristics solution from assignment 12 of CENG 571, you may copy it from owlnet. Look in the CENG671 course website.

Assignment 1-2 Truncation Error

Evaluate effect of the global truncation error by using *NX=5, 10, 20, 40, 80*. Compare history of *f* and E_R with the MOC solution. Show the convergence of *f* and E_R to the MOC solution by plotting the absolute value of the difference for *0<t<2* with *∆x* on a log-log plot. Determine the rate of convergence from the slope.

Assignment 1-3 Two Point Upstream Weighting

Do the same as assignment 2 except with the two point upstream weighting method.

Note to assignments: Comparing the recovery efficiency of the numerical methods with the MOC will not demonstrate the rate of convergence. The difference of the recovery efficiencies at $t_D=1$ is equal to the integral of the

algebraic difference between the numerical and MOC effluent fractional flow from $t_D=0$ to 1. The integral of the algebraic difference is not a norm. The integral of the **absolute** difference is a norm, the 1-norm. Attached are plots showing the convergence of numerical solution using the 1-norm from $t_D=0$ to 2. These show convergence that is of the same order as the local truncation error.

Stability

The stability of the equation will be analyzed for the linear equation assuming that f' is constant.

$$\frac{\partial S}{\partial t} + f' \frac{\partial S}{\partial x} = 0$$

The explicit (forward difference), one point upstream weighting formulation for this equation is as follows.

$$\frac{S_i^{n+1} - S_i^n}{\Delta t} + f' \left(\frac{S_i^n - S_{i-1}^n}{\Delta x} \right) = 0$$

Let $\alpha = f' \frac{\Delta t}{\Delta x}$, then

$$S_i^{n+1} = S_i^n - \alpha S_i^n + \alpha S_{i-1}^n$$

Written in matrix notation,

$$\mathbf{S}^{n+1} = (\mathbf{I} + \alpha)\mathbf{S}^n$$

$$\mathbf{I} + \alpha = \begin{bmatrix} 1-\alpha & 0 & 0 & 0 \\ \alpha & 1-\alpha & 0 & 0 \\ 0 & \alpha & 1-\alpha & 0 \\ 0 & 0 & \alpha & 1-\alpha \end{bmatrix}, \text{ect.}$$

Let **s** be a solution to the same difference equation, IC, and BC but with no round-off errors. Then the difference, **e** = **S** -**s**, has zero IC and BC.

$$\mathbf{e}^{n+1} = (\mathbf{I} + \alpha)\mathbf{e}^n$$

A round-off error at $t=t_o$ is equivalent to a nonzero initial condition for **e** at $t=t_o$, i.e., $\mathbf{e}=\mathbf{e}^o$. We wish to find out how such a error will propagate in time, i.e., will it grow in magnitude, will it oscillate? The matrix equation will be expressed in terms of its Euclidean norm. We need a few definitions and theorems (Varga 1962).

Definition 1.1. Let **x** be a (column) vector of $V_n(C)$, the n-dimensional vector space over the field of complex numbers C of column vectors **x**. Then,

$$\|\mathbf{x}\| \equiv (\mathbf{x}^*\mathbf{x})^{1/2} = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

is the ***Euclidean norm*** (or length) of **x**.

Definition 1.2. Let $A=(a_{i,j})$ be a $n \times n$ complex matrix with eigenvalues λ_i, $1 \le i \le n$. Then

$$\rho(A) \equiv \max|\lambda_i|, \quad 1 \le i \le n$$

is the ***spectral radius*** of the matrix A.

Definition 1.3. If $A=(a_{i,j})$ is an $n \times n$ complex matrix, then

$$\|A\| = \sup \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}, \quad \mathbf{x} \neq 0$$

is the ***spectral norm*** of the matrix A.
Corollary. For an arbitary $n \times n$ complex matrix, A,

$$\|A\| \ge \rho(A)$$

Theorem 1.2

$$\|A\mathbf{x}\| \le \|A\| \cdot \|\mathbf{x}\|$$

Definition 1.4. Let A be an $n \times n$ complex matrix. Then, A is **convergent** (to zero) if the sequence of matrices A, A^2, A^3, ... converges to the null matrix 0, and is **divergent** otherwise.

Theorem 1.4. If A is an $n \times n$ complex matrix, then A is convergent if and only if $\rho(A)<1$.

Theorem 1.5. ***Gerschgorin theorem*** Let $A=(a_{ij})$ be an arbitrary $n \times n$ complex matrix, and let

$$\Lambda_i \equiv \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| a_{i,j} \right|, \quad 1 \leq i \leq n$$

Then, all the eigenvalues λ of A lie in the union of the disks

$$\left| z - a_{i,i} \right| \leq \Lambda_i, \quad 1 \leq i \leq n$$

The Gerschgorin theorem can be restated as follows.

$$a_{i,i} - \Lambda_i \leq \lambda \leq a_{i,i} + \Lambda_i$$

Continuing with the matrix equation for **e**,

$$\begin{aligned} \left\| \mathbf{e}^{n+1} \right\| &= \left\| (\mathbf{I} + \alpha) \mathbf{e}^n \right\| \\ &\leq \left\| (\mathbf{I} + \alpha) \right\| \cdot \left\| \mathbf{e}^n \right\| \end{aligned}$$

Starting from $\mathbf{e}^o$,

$$\left\| \mathbf{e}^1 \right\| \leq \left\| (\mathbf{I} + \alpha) \right\| \cdot \left\| \mathbf{e}^o \right\|$$

$$\begin{aligned} \left\| \mathbf{e}^2 \right\| &\leq \left\| (\mathbf{I} + \alpha) \right\| \cdot \left\| \mathbf{e}^1 \right\| \\ &\leq \left\| (\mathbf{I} + \alpha) \right\|^2 \cdot \left\| \mathbf{e}^o \right\| \end{aligned}$$

...

$$\left\| \mathbf{e}^{n+1} \right\| \leq \left\| (\mathbf{I} + \alpha) \right\|^{n+1} \cdot \left\| \mathbf{e}^o \right\|$$

For the errors to not grow, the spectral radius of ($\mathbf{I}+\alpha$) must be less than unity. For this matrix we have

$$a_{i,i} = 1 - \alpha$$

$$\Lambda_i = \alpha$$

The bounds on the eigenvalues are

$$1 - 2\alpha \leq \lambda \leq 1$$

To keep the spectral radius less than unity, the lower limit must be greater than -1.0.

$$-1 < 1 - 2\alpha \le \lambda$$

$$f' \frac{\Delta t_D}{\Delta x_D} = \alpha < 1$$

$$\Delta t < \frac{\phi \Delta x}{\dfrac{df}{dS_w} u_T}$$

Think about what this stability condition means. The method of characteristics states that the wave velocity is equal to $f'u_T/\phi$. The numerical solution can advance only one grid block per time step with the explicit formulation i.e., the maximum velocity of a wave with the explicit formulation is $\Delta x / \Delta t$. If this ratio is less than the correct wave velocity, $f'u_T/\phi$., the numerical procedure will retard the wave front.

Assignment 1-4 Stability of the Explicit Finite Difference Procedure.

$$k_{r1} = k_{r1}^o S^{n_1}$$
$$k_{r2} = k_{r2}^o (1 - S)^{n_2}$$

$$n1 = n2 = 1.0$$
$$\alpha = 0$$
$$M = 1.0$$
$$S(x,0) = 0$$
$$f(0,t) = 1.0$$

What is the value of f' for this case? What is the exact solution? Solve for (1) saturation profile at *t=0.5* and *1.0*, (2) effluent history of *f*, and (3) recovery efficiency up to *t=2.0*. Compare with the method of characteristics (MOC) solution. Let *NX= 1.0/Δx=100* and *Δt/Δx=0.1*, 1.0, 1.01 1.02. Explain the results with respect to the truncation error and stability analysis. Does the deviation from the exact solution grow with time?

Assignment 1-5 Stability with M=10.

Do same as assignment 4 except let *M=10.0*, profiles at *t= 0.05* and *0.10*, and *Δt/Δx = 0.1, 0.5, and 1.0.*

References

1. Todd, M.R., O'Dell, P.M., and Hirasaki, G.J.: "Methods for Increased Accuracy in Numerical Reservoir Simulators", Soc. Pet. Eng. J., (December, 1972), 515-539.

2. Varga, R.S.: Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962.