

Chapter 4 Volumetric Flux

We have so far talked about Darcy's law as if it was one of the fundamental laws of nature. Here we will first describe the original experiment by Darcy in 1856. We will then generalize to cases where the flow is in more than one dimension. Finally we will identify common cases when the displacement does not follow Darcy's law.

Darcy's Experiment (Bear 1972)

In 1856, Henry Darcy investigated the flow of water in vertical homogeneous sand filters in connection with the fountains of the city of Dijon, France. Fig. 4.1 shows the experimental set-up he employed. From his experiments, Darcy concluded that the rate of flow (volume per unit time) q is (a) proportional to the constant cross-sectional area A , (b) proportional to $(h_1 - h_2)$, and inversely proportional to the length L . When combined, these conclusions give the famous Darcy formula:

$$q = KA(h_1 - h_2)/L$$

where K is the hydraulic conductivity. The pressure drop across the pack was measured by the height of the water levels in the manometer tubes. This difference in water levels is known as the *hydraulic or piezometric head* across the pack and is expressed in units of length rather than units of pressure. It is a measure of the departure from hydrostatic conditions across the pack.

Flow Potential

The hydraulic head was convenient when pressures were measured by manometers or standpipes and only one fluid is flowing. It is more useful for multidimensional, multiphase fluid flow to define a **flow potential** for each phase. (Note: There are several ways to define a potential. The potential used here is in units of pressure rather than elevation.)

$$\Phi = p - g \int_{D_o}^D \rho dD'$$

$$\Phi \cong p - \rho g (D - D_o)$$

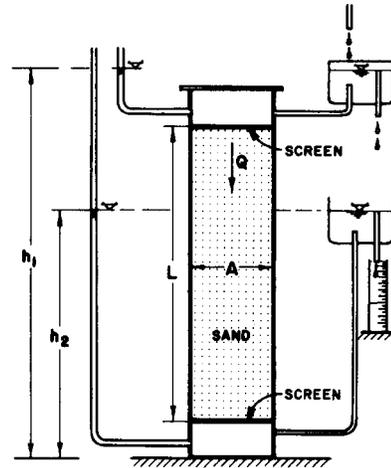


Fig. 4.1 Darcy's experiment (Bear 1972)

where ρ is the density of the fluid, g is the acceleration of gravity and D is depth with respect to some datum such as the mean sea level. The relation between the hydraulic head and the flow potential (for constant density) is

$$h = \frac{\Phi}{\rho g} + \text{constant}$$

Substituting this expression for the hydraulic head into Darcy's equation:

$$\frac{q}{A} = \frac{K}{\rho_w g} \frac{(\Phi_1 - \Phi_2)}{L}$$

The version of Darcy's law which can be applied to multiphase flow is derived by replacing the hydraulic conductivity, fluid density and acceleration of gravity by the mobility, λ . The mobility can be factored into ratio of a property of the medium and a property of the fluid, the ratio of permeability, k , divided by viscosity, μ .

$$\begin{aligned} \frac{q}{A} &= \lambda \frac{(\Phi_1 - \Phi_2)}{L} \\ &= \frac{k}{\mu} \frac{(\Phi_1 - \Phi_2)}{L} \\ \lambda &= \frac{k}{\mu} \\ K &= \frac{\rho g k}{\mu} \end{aligned}$$

Also, the flow rate per unit area is denoted as the flux, u , and the potential difference can be expressed as a gradient.

$$u = -\frac{k}{\mu} \frac{d\Phi}{dx}$$

Permeability Tensor

In three dimensions, the flux and the potential gradient are vectors and the permeability is a second order tensor.

$$\mathbf{u} = -\frac{\mathbf{k}}{\mu} \cdot \nabla \Phi$$

where in Cartesian coordinates,

$$\mathbf{k} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

It will be stated without proof that the tensor is symmetric, and there exists at least three orthogonal coordinate directions which will transform the permeability tensor into a diagonal matrix. However, if the coordinates are not aligned with these three directions, the permeability tensor will have off diagonal terms. If the medium is isotropic, the permeability tensor is diagonal with equal valued coefficients on the diagonal. In this case the permeability can be expressed as a scalar and an identity matrix. More often the permeability is isotropic in the direction of the bedding plane but anisotropic perpendicular to the bedding plane. In this case, if two coordinates are in the plane of the bedding, the permeability tensor is diagonal with the coefficients in the plane of the bedding having equal values and denoted as *horizontal permeability*, k_h , and the coefficient in the direction perpendicular to the bedding denoted as *vertical permeability*, k_v . In this case the permeability tensor will look as follows:

$$\mathbf{k} = \begin{bmatrix} k_h & 0 & 0 \\ 0 & k_h & 0 \\ 0 & 0 & k_v \end{bmatrix}$$

Marine deposits usually have the bedding plane parallel to the sand strata. However, cross-bedded deposits have bedding planes that are not parallel to the sand strata and non zero off diagonal terms will exist.

Permeability Micro-Heterogeneity

Natural sediments are not homogeneous even on a micro-scale of millimeters. Inspection of a consolidated sandstone rock sample shows laminations that parallel the bedding plane of the sediment. These heterogeneities are of too small scale to describe individually in a reservoir simulation. It is necessary to average the reservoir properties over the dimensions of a simulation grid block. These small scale heterogeneities in permeability can be described as in parallel or perpendicular to the direction of flow.

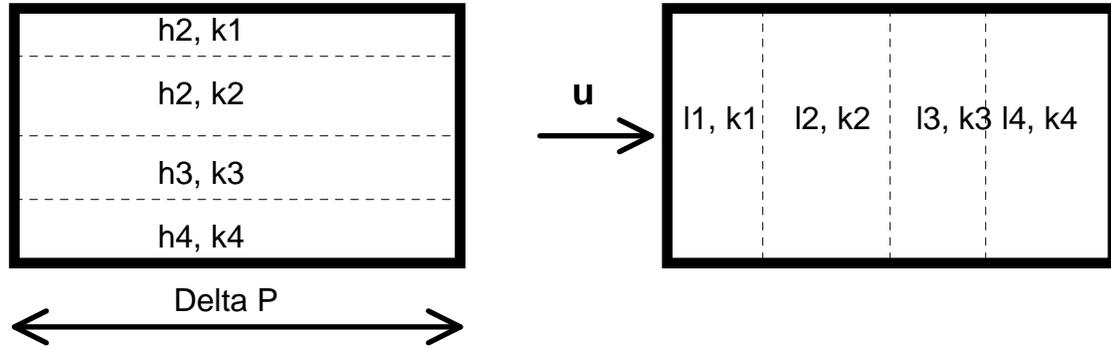


Fig. 4.2 Layering parallel and perpendicular (in series) to the direction of flow.

Composite permeability for layers in parallel

When the flow is parallel to the bedding, all layers have a common pressure drop and length. The flow rate and cross sectional area is the sum of the individual layers.

$$q = \frac{w}{\mu} \frac{\Delta\Phi}{L} \sum_{n=1}^N k_n h_n$$

$$A = w \sum_{n=1}^N h_n$$

$$h = \sum_{n=1}^N h_n$$

If we express the composite permeability of this system in parallel as an average permeability, we have

$$\bar{k} = \frac{\sum_{n=1}^N k_n h_n}{\sum_{n=1}^N h_n}$$

This average is called an arithmetic average. This average is dominated by the higher permeability layers.

Composite Permeability for Layers in Series

Now consider the system in series. Each layer has a common flux, u , and cross sectional area, A . However, each layer has a distinct length, ℓ_n and permeability, k_n . The pressure drop over the length of the composite system is the sum of the pressure drops over the individual layers. Also, the length of the composite system is the sum of the individual layer thicknesses.

$$u = \frac{\bar{k} \Delta p}{\mu L}$$

$$\bar{k} = \mu u \frac{L}{\Delta p}$$

$$L = \sum_{n=1}^N l_n$$

$$\Delta p = \sum_{n=1}^N \Delta p_n$$

$$\Delta p_n = \mu u \frac{l_n}{k_n}$$

$$\bar{k} = \frac{\sum_{n=1}^N l_n}{\sum_{n=1}^N \frac{l_n}{k_n}}$$

This average is sometimes called a harmonic average. This average is dominated by the lower permeability layers.

Permeability Anisotropy

We mentioned earlier that the permeability may be greater parallel to the bedding than perpendicular to the bedding. The permeability tensor will still be diagonal if one coordinate direction is aligned with the perpendicular direction to the bedding. However, if a formation is cross-bedded it may not be possible to align the coordinates with the bedding because now the bedding planes will no longer be parallel with the reservoir boundaries. In such a case the permeability tensor will have off diagonal terms and the flux vector will not be collinear with the potential gradient. The off diagonal terms of the permeability tensor can be calculated from the definition of a second order Cartesian tensor. Let us consider the transformation of the coordinate frame by an arbitrary rigid rotation. Fig. 4.3 shows the original frame $O123$ in which the coordinates of a point P are x_1, x_2, x_3 and the new frame $O\underline{1}\underline{2}\underline{3}$ in which they are $\underline{x}_1, \underline{x}_2, \underline{x}_3$. Let l_{ij} be the cosine of the angle between O_i and O_j , $i, j=1,2,3$. Thus l_{1j}, l_{2j}, l_{3j} are the directional cosines in the old system, i.e., they are the projection of O_j on $O123$. By definition (Aris 1962) of a Cartesian vector, the new coordinate \underline{x}_j is the length of the projection of OP on the axis O_j , i.e.,

$$\bar{x}_j = l_{1j}x_1 + l_{2j}x_2 + l_{3j}x_3$$

The definition (Aris 1962) of a second order Cartesian tensor is an entity having nine components A_{ij} , $i,j=1,2,3$, in the Cartesian coordinates $O123$ which on rotation of the coordinates to $O\underline{1}\underline{2}\underline{3}$ become

$$\begin{aligned}\bar{A}_{pq} &= l_{ip} l_{jq} A_{ij} \\ &= l'_{pi} A_{ij} l_{jq}\end{aligned}$$

Lets consider a simpler case of anisotropy in two dimensions. Suppose that as illustrated in Fig. 4.4, the coordinate directions of x_1 and x_2 coincide with the principal direction of the permeability tensor so that $A_{12}=A_{21}=0$. Now suppose that the coordinate system is rotated to $\underline{x}_1, \underline{x}_2$ by a rotation about the x_3 axis. The directional cosines of this rotation is given by the following matrix.

$$l_{ij} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This rotation will transform an anisotropic, diagonal tensor in the x_1, x_2 coordinate system to a tensor with off diagonal terms in the $\underline{x}_1, \underline{x}_2$ coordinate system. The off diagonal terms can be negative.

Another way to view the effect of anisotropy is to compare the directions of the potential gradient and flux vectors. Suppose that the coordinates x_1, x_2 are aligned with the principal directions of the permeability tensor and potential gradient divided by viscosity is given as a unit vector inclined at an angle θ from the x_1 axis. The components of the flux are then:

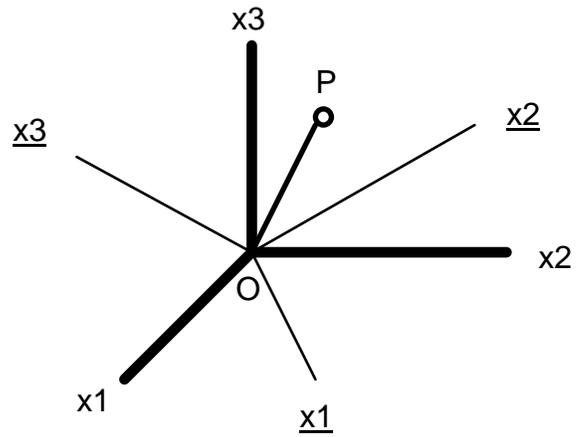


Fig. 4.3 Rotation of a Cartesian coordinate system from x_j to \underline{x}_i .

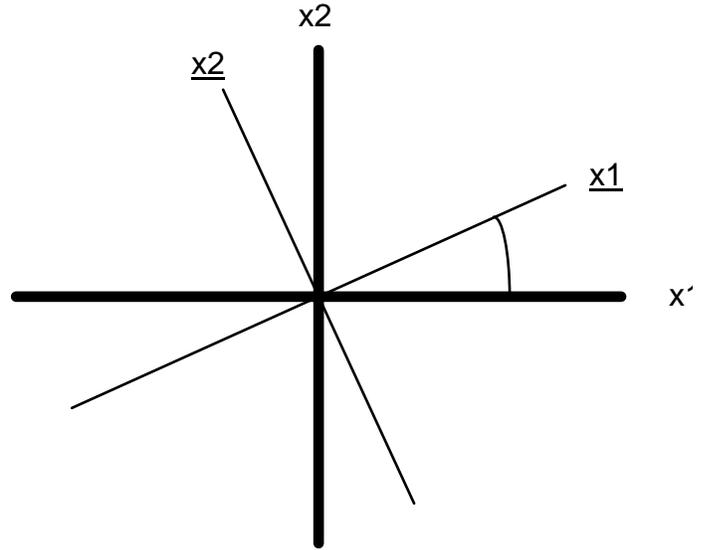
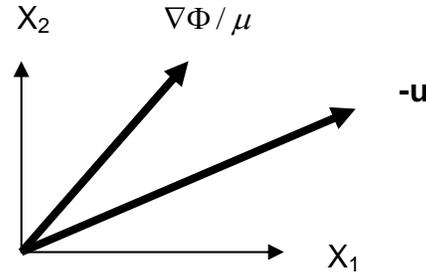


Fig. 4.4 Rotation of the x_1, x_2 coordinates about X_3 by and angle θ .

$$\mathbf{u} = -\mathbf{k} \left(\frac{\nabla \Phi}{\mu} \right)$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = - \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

$$= \begin{bmatrix} -k_{11} \cos \theta \\ -k_{22} \sin \theta \end{bmatrix}$$



The direction of the flux θ_u can be compared with the direction of the potential gradient to determine the departure from collinearity.

$$\theta_u - \pi = \arccos \left[\frac{k_{11} \cos \theta}{\sqrt{(k_{11} \cos \theta)^2 + (k_{22} \sin \theta)^2}} \right]$$

Assignment 4.1 Anisotropic Tensor

Suppose a system is layered with 1/10 of the thickness having a permeability of 1 md and 9/10 of the thickness having a permeability of 10 md.

- What is the average permeability parallel and perpendicular to the bedding?
- Suppose the original coordinate system has x_1 parallel to the direction of bedding and x_2 perpendicular to the direction of bedding. Let $\underline{x}_1, \underline{x}_2$ be another coordinate system rotated an angle θ from x_1, x_2 . Derive expressions for \underline{k}_{ij} in terms of k_{11}, k_{22} , and θ .
- Plot components of \underline{k}_{ij} for $0 \leq \theta \leq \pi/2$.
- Suppose the potential gradient is given by the unit vector in the direction θ from x_1 . Calculate and plot the direction $\alpha - \pi$ of the (negative) flux for $0 \leq \theta \leq \pi/2$. Also, plot the unit slope line which would result if the potential gradient and flux were collinear.

Darcy's Law from Momentum Balance

So far we have generalized Darcy's law from observations made in the 19th century. Another approach that has been made is to derive Darcy's law from first principals. Here we will use the latter approach just in enough detail to see what we are neglecting in Darcy's Law. The equation of motion or momentum balance of a continuum fluid is (Bird, Stewart, and Lightfoot 1960)

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + [\nabla \cdot \rho \mathbf{v} \mathbf{v}] = -[\nabla \cdot \boldsymbol{\tau}] - \nabla p + \rho \mathbf{g}$$

The left hand side terms are the rate of increase of momentum per unit volume plus the rate of momentum gain by convection per unit volume. These are the

inertial terms that are neglected in Darcy's law. The inertial terms become important in flowing systems with high velocity, high permeability, and/or low viscosity. We will neglect inertial terms for the moment but return to it later. The term in brackets is the rate of momentum gain by viscous transfer per unit volume. Since the shear stress τ is a tensor, the divergence of the shear stress is a vector. The momentum balance given above apply only in the pore spaces of the porous medium and the pore walls are boundaries for the domain of the equation. To derive a momentum balance that will apply for a volume of the porous medium that is large compared to the size of individual pores, we must integrate the equation over the volume and divide by the volume to express the momentum balance in terms of average quantities. The volume average of the pressure gradient and buoyancy terms will recover the same expression. The average of the viscous transfer term can be expressed as follows (Aris 1962).

$$\frac{\int_{V_\phi} \nabla \cdot \tau \, dV}{\int_{V_\phi} dV} = \frac{\int_{S_\phi} \tau \cdot \mathbf{n} \, dS}{\int_{V_\phi} dV} + \frac{\int_{S_{ext}} \tau \cdot \mathbf{n} \, dS}{\int_{V_\phi} dV}$$

$$\approx \tau \cdot \mathbf{n} \left(\frac{1}{d_p} + \frac{1}{L} \right)$$

The integral is over a volume of the pore space enclosed by L^3 that has internal surfaces or pore walls denoted by S_ϕ and the external surfaces denoted by S_{ext} . We saw earlier that the ratio of the pore wall surface to the pore volume is inversely proportional to the grain diameter. The ratio of the external surface to the pore volume is inversely proportional to L . Thus as L becomes large compared to d_p , the second integral can be neglected. The dot product of the stress with the pore walls can be approximated as follows;

$$\tau \cdot \mathbf{n} \approx -\mu \frac{\partial v_x}{\partial y}$$

$$\approx -\frac{\mu \bar{v}}{d_p}$$

Thus the momentum balance neglecting inertial terms can be expressed as follows:

$$0 \approx -\frac{\mu \bar{v}}{d_p^2} - \nabla p + \rho \mathbf{g}$$

$$\bar{v} \approx -\frac{d_p^2}{\mu} (\nabla p - \rho \mathbf{g})$$

We have derived a relationship which states that the average velocity in the pore space is proportional to a potential gradient consisting of a pressure and gravity term and the coefficient is proportional to the square of a characteristic pore or grain dimension and is inversely proportional to the fluid viscosity.

Non-Darcy Flow

Darcy's law applies in most cases but cases where it does not apply should be recognized. Alternative models have been developed in most cases where Darcy's law does not apply.

High Reynolds Number Flow

We derived the Blake-Kozeny equation while neglecting inertial terms. Fig. 4.5 is a plot of friction factor versus Reynolds number for a packed bed (Bird, Stewart, and Lightfoot 1962). G_0 is the mass flux, ρv_0 . This figure shows that the Blake-Kozeny model which obeys Darcy's law will apply for Reynolds number less than 10 but inertial effects must be included for higher Reynolds numbers. The Ergun equation combines the low Reynolds number Blake-Kozeny model with the high Reynolds number Burke-Plummer model. The result works for both regimes.

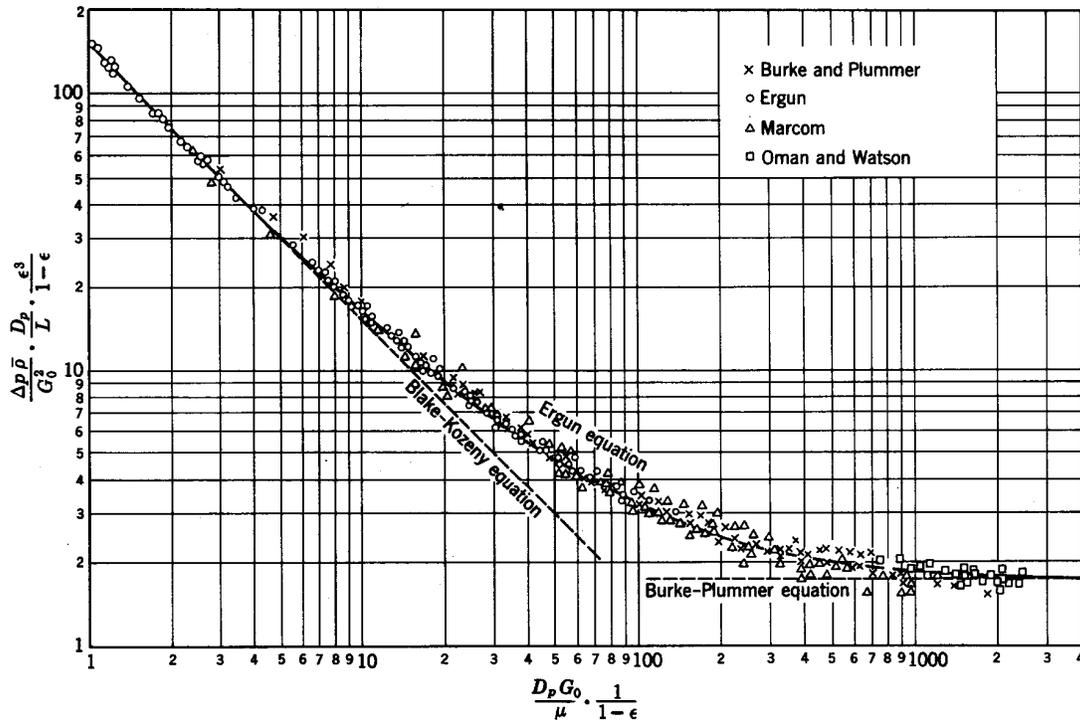


Fig. 4.5 Friction factor versus Reynolds number for flow in packed bed (Bird, Stewart, and Lightfoot 1960)

Low Pressure Gas

At low pressures the mean free path of the gas molecules in the pores may be less than the pore diameter. This will give the appearance of "slip flow" and the permeability will be dependent of the absolute pressure. This effect is called the Klinkenberg effect and the pressure independent permeability is estimated by extrapolating to infinite pressure. A model for the pressure dependent permeability is as follows.

$$k = k_{\infty} \left(1 + \frac{b}{p} \right)$$

Compressible gas

Another factor at low gas pressures is the large compressibility which will result in the flux changing in even a one dimensional system. This is not a problem with the differential form of Darcy's law but with finite difference approximation or when using the integrated form of Darcy's law, an average density must be used. This can be avoided by using the pressure squared as the dependent variable if the pressure is low enough for the ideal gas law to be obeyed.

$$\begin{aligned} \rho u &= -\frac{p}{MRT} \frac{k}{\mu} \frac{\partial p}{\partial x} \\ &= -\frac{k}{2MRT\mu} \frac{\partial p^2}{\partial x} \\ &= -\frac{k}{2MRT\mu} \frac{(\Delta p^2)}{L} \end{aligned}$$

If the gas does not obey the ideal gas law or the viscosity changes with pressure, the real gas pseudo pressure can be used.

$$m = \frac{1}{2MRT} \int_{p_o}^p \frac{p' dp'}{z\mu}$$

$$\rho \mathbf{u} = -k \nabla m$$

Complex Fluids

Darcy's law departs from a linear relation between the potential gradient and the flux when the viscosity or the relative permeability depends on the flux or pressure gradient. Examples are polymer solutions and foams.

References

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Bird, R. B., Stewart, W. E., and Lightfoot, E. N.: Transport Phenomena, John Wiley & Sons, New York, 1960.

Bear, J.: Dynamics of Fluids in Porous Media, Dover Publications, Inc., New York, 1972.