Chapter 4 - The Kinematics of Fluid Motion

Particle paths and material derivatives
Streamlines
Streaklines
Dilatation
Reynolds' transport theorem
Conservation of mass and the equation of continuity
Deformation and rate of strain
Physical interpretation of the deformation tensor
Principal axis of deformation
Vorticity, vortex lines, and tubes

Reading assignment: Chapter 4 of Aris

Kinematics is the study of motion without regard to the forces that bring about the motion. Already, we have described how rigid body motion is described by its translation and rotation. Also, the divergence and curl of the field and values on boundaries can describe a vector field. Here we will consider the motion of a fluid as microscopic or macroscopic bodies that translate, rotate, and deform with time. We treat fluids as a continuum such that the fluid identified to be at a specific point in space at one time with neighboring fluid will be at another specific point in space at a later time with the same neighbors, with the exception of certain bifurcations. This identification of the fluid occupying a point in space requires that the motion is deterministic rather than stochastic, i.e., random motions such as diffusion and turbulence are not described. Central to the kinematics of fluid motion is the concept of convection or following the motion of a "particle" of fluid.

Particle paths and material derivatives

Fluid motion will be described as the motion of a "particle" that occupies a point in space. At some time, say $t=0$, a fluid particle is at a position $\xi = (\xi_1, \xi_2, \xi_3)$ and at a later time the same particle is at a position **x**. The motion of the particle that occupied this original position is described as follows.

$$\mathbf{x} = \mathbf{x}(\xi, t) \text{ or } x_i = x_i(\xi_1, \xi_2, \xi_3, t)$$

The initial coordinates ξ of a particle will be referred to as the *material coordinates* of the particles and, when convenient, the particle itself may be called the particle ξ. The terms *convected* and *Lagrangian* coordinates are also used. The *spatial coordinates* **x** of the particle may be referred to as its *position* or *place*. It will be assumed that the motion is continuous, single valued and the previous equation can be inverted to give the initial position or material coordinates of the particle which is at any position **x** at time t; i.e.,

$$\xi = \xi(\mathbf{x}, t) \ \ or \ \ \xi_i = \xi_i(x_1, x_2, x_3, t)$$

are also continuous and single valued. Physically this means that a continuous arc of particles does not break up during the motion or that the particles in the neighborhood of a given particle continue in its neighborhood during the motion. The single valuedness of the equations mean that a particle cannot split up and occupy two places nor can two distinct particles occupy the same place. Exceptions to these requirements may be allowed on a finite number of singular surfaces, lines or points, as for example a fluid divides around an obstacle. It is shown in Appendix B that a necessary and sufficient condition for the inverse functions to exist is that the Jacobian

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)}$$

should not vanish.

The transformation $\mathbf{x}=\mathbf{x}(\xi,t)$ may be looked at as the parametric equation of a curve in space with t as the parameter. The curve goes through the point ξ, corresponding to the parameter t=0, and these curves are the *particle paths*. Any property of the fluid may be followed along the particle path. For example, we may be given the density in the neighborhood of a particle as a function $\rho(\xi,t)$, meaning that for any prescribed particle ξ we have the density as a function of time, that is, the density that an observer riding on the particle would see. (Position itself is a “property” in this general sense so that the equations of the particle path are of this form.) This *material* description of the change of some property, say $\Im(\xi,t)$, can be changed to a spatial description $\Im(\mathbf{x},t)$.

$$\Im(\mathbf{x},t) = \Im\left[\xi(\mathbf{x},t),t\right]$$

Physically this says that the value of the property at position **x** at time t is the value appropriate to the particle that is at **x** at time t. Conversely, the material description can be derived from the spatial one.

$$\Im(\xi,t) = \Im\left[\xi(\mathbf{x},t),t\right]$$

meaning that the value as seen by the particle at time t is the value at the position it occupies at that time.

Associated with these two descriptions are two derivatives with respect to time. We shall denote them by

$$\frac{\partial}{\partial t} \equiv \left(\frac{\partial}{\partial t}\right)_x = \text{derivative with respect to time keeping } \mathbf{x} \text{ constant.}$$

and

$$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t}\right)_\xi = \text{derivative with respect to time keeping } \xi \text{ constant}$$

Thus $\partial\Im/\partial t$ is the rate of change of $\Im$ as observed at a fixed point **x**, whereas $D\Im/Dt$ is the rate of changed as observed when moving with the particle, i.e., for a fixed value of ξ. The latter we call the *material derivative*. It is also called the convected, convective, or substantial derivative and often denoted by *D/Dt*. In particular the material derivative of the position of a particle is it velocity. Thus putting $\Im = x_i$, we have

$$v_i = \frac{Dx_i}{Dt} = \frac{\partial}{\partial t} x_i(\xi_1, \xi_2, \xi_3, t)$$

or .

$$\mathbf{v} = \frac{D\mathbf{x}}{Dt}$$

This allows us to establish a connection between the two derivatives, for

$$\begin{aligned}
\frac{D\Im}{Dt} &= \frac{\partial}{\partial t}\Im(\xi,t) = \frac{\partial}{\partial t}\Im\left[x(\xi,t),t\right] \\
&= \left(\frac{\partial \Im}{\partial t}\right)_x + \frac{\partial \Im}{\partial x_i}\left(\frac{\partial x_i}{\partial t}\right)_\xi \\
&= \frac{\partial \Im}{\partial t} + v_i \frac{\partial \Im}{\partial x_i} \\
&= \frac{\partial \Im}{\partial t} + (\mathbf{v}\bullet\nabla)\Im
\end{aligned}$$

Streamlines

We now have a formal definition of the velocity field as a material derivative of the position of a particle.

$$\mathbf{v}(\mathbf{x},t) = \frac{D\mathbf{x}(\xi,t)}{Dt} = \frac{\partial \mathbf{x}(\xi,t)}{\partial t}$$

The field lines of the velocity field are called *streamlines*; they are the solutions of the three simultaneous equations

$$\frac{d\mathbf{x}}{ds} = \mathbf{v}(\mathbf{x},t)$$

where *s* is a parameter along the streamline. This parameter *s* is not to be confused with the time, for in the above equation *t* is held fixed while the equations are integrated, and the resulting curves are the streamlines *at the instant t*. These may vary from instant to instant and in general will not coincide with the particle paths.

To obtain the *particle paths* from the velocity field we have to follow the motion of each particle. This means that we have to solve the differential equations

$$\frac{D\mathbf{x}}{Dt} = \mathbf{v}(\mathbf{x}, t)$$

subject to $\mathbf{x}=\xi$ at t=0. Time is the parameter along the particle path. Thus the particle path is the trajectory taken by a particle.

The flow is called *steady* if the velocity components are independent of time. For steady flows, the parameter s along the streamlines may be taken to be t and the streamlines and particle paths will coincide. The converse does not follow as there are unsteady flows for which the streamlines and particle paths coincide.

If C is a closed curve in the region of flow, the streamlines through every point of C generate a surface known as a *stream tube*. Let S be a surface with C as the bounding curve, then

$$\iint_S \mathbf{v} \bullet \mathbf{n} \, dS$$

is known as the *strength of the stream tube* at its cross-section S.

The *acceleration* or the rate of change of velocity is defined as

$$\mathbf{a} = \frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \bullet \nabla)\mathbf{v}$$
$$a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}$$

Notice that in steady flow this does not vanish but reduces to

$$\mathbf{a} = (\mathbf{v} \bullet \nabla)\mathbf{v} \quad \text{for steady flow} \, .$$

Even in steady flow other than a constant translation, a fluid particle will accelerate if it changes direction to go around an obstacle or if it increases its speed to pass through a constriction.

Streaklines

The name streakline is applied to the curve traced out by a plume of smoke or dye, which is continuously injected at a fixed point but does not diffuse. Thus at time t the streakline through a fixed-point $\mathbf{y}$ is a curve going from $\mathbf{y}$ to $\mathbf{x}(\mathbf{y}, t)$, the position reached by the particle which was at $\mathbf{y}$ at time t=0. A particle is on the streakline if it passed the fixed-point $\mathbf{y}$ at some time between 0 and t. If this time was t', then the material coordinates of the particle would be given by ξ

= $\xi(\mathbf{y}, t')$. However, at time t this particle is at $\mathbf{x} = \mathbf{x}(\xi, t)$ so that the equation of the streakline at time t is given by

$$\mathbf{x} = \mathbf{x}\left[\xi(\mathbf{y},\ t'), t\right],$$

where the parameter t' along it lies in the interval $0 \le t' \le t$. If we regard the motion as having been proceeding for all time, then the origin of time is arbitrary and t' can take negative values $-\infty \le t' \le t$.

The flow field illustrated in 4.13 by Aris is assigned as an exercise.

Dilatation

We noticed earlier that if the coordinate system is changed from coordinates ξ to coordinates $\mathbf{x}$, then the element of volume changes by the formula

$$dV = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} d\xi_1\, d\xi_2\, d\xi_3 = J\, dV_0$$

If we think of ξ as the material coordinates, they are the Cartesian coordinates at $t = 0$, so that $d\xi_1\ d\xi_2\ d\xi_3$ is the volume dV_o of an elementary rectangular parallelepiped. Consider this elementary parallelepiped about a given point ξ at the initial instant. By the motion this parallelepiped is moved and distorted but because the motion is continuous it cannot break up and so at some time t is some neighborhood of the point $\mathbf{x} = \mathbf{x}(\xi, t)$. By the above equation, its volume is $dV = J\ dV_o$ and hence

$$J = \frac{dV}{dV_o} = \text{ratio of an elementary material volume to its initial volume}.$$

It is called the *dilation* or *expansion*. The assumption that $\mathbf{x} = \mathbf{x}(\xi, t)$ can be inverted to give $\xi = \xi(\mathbf{x}, t)$, and vice versa, is equivalent to requiring that neither J nor J^{-1} vanish. Thus,

$$0 < J < \infty$$

We can now ask how the dilation changes as we follow the motion. To answer this we calculate the material derivative DJ/Dt. However,

$$J = \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3} = \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{vmatrix}$$

Now

$$\frac{D}{Dt}\left(\frac{\partial x_i}{\partial \xi_j}\right) = \frac{\partial}{\partial \xi_j} \frac{Dx_i}{Dt} = \frac{\partial v_i}{\partial \xi_j}$$

for *D/Dt* is differentiation with ξ constant so that the order can be interchanged. Now if we regard v_i as a function of x_1, x_2, x_3,

$$\frac{\partial v_i}{\partial \xi_j} = \frac{\partial v_i}{\partial x_m} \frac{\partial x_m}{\partial \xi_j}$$

The above relation can now be applied to differentiation of the Jacobian.

$$\begin{aligned}
\frac{DJ}{Dt} &= \varepsilon_{ijk} \frac{D}{Dt}\left(\frac{\partial x_i}{\partial \xi_1}\right) \frac{\partial x_j}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3} + \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{D}{Dt}\left(\frac{\partial x_j}{\partial \xi_2}\right) \frac{\partial x_k}{\partial \xi_3} + \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{D}{Dt}\left(\frac{\partial x_k}{\partial \xi_3}\right) \\
&= \varepsilon_{ijk} \frac{\partial v_i}{\partial x_m} \frac{\partial x_m}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3} + \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial v_j}{\partial x_m} \frac{\partial x_m}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3} + \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{\partial v_k}{\partial x_m} \frac{\partial x_m}{\partial \xi_3} \\
&= \varepsilon_{ijk} \frac{\partial v_1}{\partial x_1} \frac{\partial x_i}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3} + \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial v_2}{\partial x_2} \frac{\partial x_j}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3} + \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{\partial v_3}{\partial x_3} \frac{\partial x_k}{\partial \xi_3} \\
&= \frac{\partial v_1}{\partial x_1} J + \frac{\partial v_2}{\partial x_2} J + \frac{\partial v_3}{\partial x_3} J \\
&= \left(\nabla \bullet \mathbf{v}\right) J
\end{aligned}$$

where we made use of the property of the determinant that the determinant of a matrix with repeated rows is zero. Thus,

$$\frac{D(\ln J)}{Dt} = \nabla \bullet \mathbf{v}$$

We thus have an important physical meaning for the divergence of the velocity field. It is the relative rate of dilation following a particle path. It is evident that for an incompressible fluid motion,

$$\nabla \bullet \mathbf{v} = 0 \quad \text{for incompressible fluid motion}.$$

Reynolds' transport theorem

An important kinematical theorem can be derived from the expression for the material derivative of the Jacobian. It is due to Reynolds and concerns the rate of change not of an infinitesimal element of volume but any volume integral. Let $\Im(\mathbf{x}, t)$ be any function and *V(t)* be a closed volume moving with the fluid, that is consisting of the same fluid particles. Then

$$F(t) = \iiint_{V(t)} \Im(\mathbf{x},\ t)\, dV$$

is a function of *t* that can be calculated. We are interested in its material derivative *DF/Dt*. Now the integral is over the varying volume *V(t)* so we cannot take the differentiation through the integral sign. If, however, the integration were with respect to a volume in ξ -space it would be possible to interchange differentiation and integration since *D/Dt* is differentiation with respect to *t* keeping ξ constant. The transformation $\mathbf{x} = \mathbf{x}(\xi, t)$, $dV = J\, dV_o$ allows us to do just this, for *V(t)* has been defined as a moving material volume and so come from some fixed volume V_o at $t = 0$. Thus

$$\begin{aligned}
\frac{d}{dt}\iiint_{V(t)} \Im(\mathbf{x},\ t)\, dV &= \frac{d}{dt}\iiint_{V_o} \Im\left[\mathbf{x}(\xi,\ t), t\right] J\, dV_o \\
&= \iiint_{V_o}\left(\frac{D\Im}{Dt} J + \Im\frac{DJ}{Dt}\right) dV_o \\
&= \iiint_{V_o}\left(\frac{D\Im}{Dt} + \Im(\nabla \bullet \mathbf{v})\right) J\, dV_o \\
&= \iiint_{V(t)}\left(\frac{D\Im}{Dt} + \Im(\nabla \bullet \mathbf{v})\right) dV
\end{aligned}$$

Since $D/Dt = (\partial/\partial t) + \mathbf{v}\bullet\nabla$ we can express this formula into a number of different forms. Substituting for the material derivative and collecting the gradient terms gives

$$\begin{aligned}
\frac{d}{dt}\iiint_{V(t)} \Im(\mathbf{x}, t)\, dV &= \iiint_{V(t)}\left(\frac{\partial\Im}{\partial t} + \mathbf{v}\bullet\nabla\Im + \Im(\nabla \bullet \mathbf{v})\right) dV \\
&= \iiint_{V(t)}\left(\frac{\partial\Im}{\partial t} + \nabla\bullet(\Im\,\mathbf{v})\right) dV
\end{aligned}$$

Now applying Green's theorem to the second integral we have

$$\frac{d}{dt}\iiint_{V(t)} \Im(\mathbf{x},t)\,dV = \iiint_{V(t)} \frac{\partial \Im}{\partial t}\,dV + \oiint_{S(t)} (\Im\,\mathbf{v}) \bullet \mathbf{n}\,dS$$

where *S(t)* is the bounding surface of *V(t).* This admits of an immediate physical picture for it says that the rate of change of the integral of $\Im$ within the moving volume is the integral of the rate of change of $\Im$ at a point plus the net flow of $\Im$ over the bounding surface. $\Im$ can be any scalar or tensor component, so that this is a kinematical result of wide application. It is going to be the basis for the conservation of mass, momentum, energy, and species. This approach to the conservation equations differs from the approach taken by Bird, Stewart, and Lightfoot. They perform a balance on a fixed volume of space and explicitly account for the convective flux across the boundaries.

Conservation of mass and the equation of continuity

Although the idea of mass is not a kinematical one, it is convenient to introduce it here and to obtain the continuity equation. Let $\rho(\mathbf{x}, t)$ be the mass per unit volume of a homogeneous fluid at position **x** and time *t*. Then the mass of any finite material volume *V(t)* is

$$m = \iiint_{V(t)} \rho(\mathbf{x},t)\,dV \,.$$

If *V* is a material volume, that is, if it is composed of the same particles, and there are no sources or sinks in the medium we take it as a principle that the mass does not change. By inserting $\Im = \rho$ in Reynolds' transport theorem we have

$$\begin{aligned}
\frac{dm}{dt} &= \iiint_{V(t)} \left\{ \frac{D\rho}{Dt} + \rho(\nabla \bullet \mathbf{v}) \right\} dV \\
&= \iiint_{V(t)} \left\{ \frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \quad \mathbf{v}) \right\} dV \\
&= 0
\end{aligned}$$

This is true for an arbitrary material volume and hence the integrand itself must vanish everywhere. It follows that

$$\frac{D\rho}{Dt} + \rho(\nabla \bullet \mathbf{v}) = \frac{\partial \rho}{\partial t} + \nabla \bullet (\rho\,\mathbf{v}) = 0$$

which is the equation of continuity.

A fluid for which the density ρ is constant is called *incompressible*. In this case the equation of continuity becomes

$$\nabla \bullet \mathbf{v} = 0 \;\; \text{incompressible flow}$$

and the motion is isochoric or the velocity field solenoidal.

Combining the equation of continuity with Reynolds' transport theorem for a function $\Im = \rho F$ we have

$$\frac{d}{dt}\iiint_{V(t)} \rho F\, dV = \iiint_{V(t)} \left\{ \frac{D}{Dt}(\rho F) + \rho F\,(\nabla \bullet \mathbf{v}) \right\} dV$$

$$= \iiint_{V(t)} \left\{ \rho \frac{DF}{Dt} + F\,(\frac{D\rho}{Dt} + \rho \nabla \bullet \mathbf{v}) \right\} dV$$

$$= \iiint_{V(t)} \rho \frac{DF}{Dt}\, dV$$

$$= \iiint_{V(t)} (\text{sources})\, dV$$

Thus

$$\iiint_{V(t)} \left[\rho \frac{DF}{Dt} - (\text{sources}) \right] dV = 0, \quad \text{for any } V(t)$$

$$\text{thus,} \quad \rho \frac{DF}{Dt} - (\text{sources}) = 0$$

This equation is useful for deriving the conservation equation of a quantity that is expressed as specific to a unit of mass, e.g., specific internal energy and species mass fraction.

Deformation and rate of strain

The motion of fluids differs from that of rigid bodies in the deformation or strain that occurs with motion. Material coordinates give a reference frame for this deformation or strain.

Consider two nearby points *P* and *Q* with material coordinates ξ and $\xi + d\xi$. At time *t* they are to be found at $\mathbf{x}(\xi, t)$ and $\mathbf{x}(\xi + d\xi, t)$. Now

$$x_i(\xi + d\xi, t) = x_i(\xi, t) + \frac{\partial x_i}{\partial \xi_j} d\xi_j + O(d^2)$$

where $O(d^2)$ represents terms of order $d\xi^2$ and higher which will be neglected from this point onward. Thus the small displacement vector $d\xi$ has now become

$$d\mathbf{x} = \mathbf{x}(\xi + d\xi, t) - \mathbf{x}(\xi, t)$$

where

$$dx_i = \frac{\partial x_i}{\partial \xi_j} d\xi_j \ .$$

It is clear from the quotient rule (since $d\xi$ is arbitrary) that the nine quantities $\partial x_i/\partial \xi_j$ are the components of a tensor. It may be called the *displacement gradient tensor* and is basic to the theories of elasticity and rheology. For fluid motion, its material derivative is of more direct application and we will concentrate on this.

If $\mathbf{v} = D\mathbf{x}/Dt$ is the velocity, the relative velocity of two particles ξ and $\xi + d\xi$ has components

$$dv_i = \frac{\partial v_i}{\partial \xi_k} d\xi_k = \frac{D}{Dt}\left(\frac{\partial x_i}{\partial \xi_k}\right) d\xi_k$$

However, by inverting the above relation, we have

$$dv_i = \frac{\partial v_i}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j} dx_j = \frac{\partial v_i}{\partial x_j} dx_j$$

expressing the relative velocity in terms of current position. Again it is evident that the $(\partial v_i/\partial x_j)$ are components of a tensor, the *velocity gradient tensor*, for which we need to obtain a sound physical feeling.

We first observe that if the motion is a rigid body translation with a constant velocity **u**,

$$\mathbf{x} = \xi + \mathbf{u}t$$

and the velocity gradient tensor vanishes identically. Secondly, the velocity gradient tensor can be written as the sum of symmetric and antisymmetric parts,

$$\begin{aligned}
\frac{\partial v_i}{\partial x_j} &\equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \\
&= e_{ij} + \Omega_{ij} \\
&\text{or} \\
\nabla \mathbf{v} &= \mathbf{e} + \mathbf{\Omega}
\end{aligned}$$

We have seen that a relative velocity dv_i related to the relative position dx_j by an antisymmetric tensor Ω_{ij}, i.e., $dv_i = \Omega_{ij}\, dx_j$, represents a rigid body rotation with angular velocity $\boldsymbol{\omega} = -\text{vec}\ \mathbf{\Omega}$. In this case

$$\omega_i = -\frac{1}{2}\varepsilon_{ijk}\,\Omega_{jk} = \frac{1}{2}\varepsilon_{ijk}\,\frac{\partial v_k}{\partial x_j}$$

or

$$\omega = \frac{1}{2}\nabla\times\mathbf{v}$$

Thus the antisymmetric part of the velocity gradient tensor corresponds to rigid body rotation, and, if the motion is a rigid one (composed of a translation plus a rotation), the symmetric part of the velocity gradient tensor will vanish. For this reason the tensor e_{ij} is called the *deformation* or *rate of strain* tensor and its vanishing is necessary and sufficient for the motion to be without deformation, that is, rigid.

Physical interpretation of the (rate of) deformation tensor

The (rate of) deformation tensor is what distinguishes fluid motion from rigid body motion. Recall that a rigid body is one in which the relative distance between two points in the body does not change. We show here that the (rate of) deformation tensor describes the rate of change of the relative distance between two particles in a fluid. Also, it describes the rate of change of the angle between three particles in the fluid.

First we will see how the distance between two material points change during the motion. The length of an infinitesimal line segment from P to Q is ds, where

$$ds^2 = dx_i\,dx_i = \frac{\partial x_i}{\partial \xi_j}\frac{\partial x_i}{\partial \xi_k}\,d\xi_j\,d\xi_k \ .$$

now P and Q are the material particles ξ and ξ + dξ so that $d\xi_j$ *and* $d\xi_k$ do not change during the motion. Also, recall that $Dx_i/Dt = v_i$. Thus

$$\frac{D}{Dt}(ds^2) = \left(\frac{\partial v_i}{\partial \xi_j}\frac{\partial x_i}{\partial \xi_k} + \frac{\partial x_i}{\partial \xi_j}\frac{\partial v_i}{\partial \xi_k}\right)d\xi_j\,d\xi_k = 2\,\frac{\partial v_i}{\partial \xi_j}\frac{\partial x_i}{\partial \xi_k}\,d\xi_j\,d\xi_k$$

by symmetry. However,

$$\frac{\partial v_i}{\partial \xi_j}d\xi_j = dv_i = \frac{\partial v_i}{\partial x_j}dx_j \quad \text{and} \quad \frac{\partial x_i}{\partial \xi_k}d\xi_k = dx_i$$

$$\text{since} \quad \mathbf{v} = \mathbf{v}\left[\mathbf{x}(\xi)\right] \quad \text{and} \quad \mathbf{x} = \mathbf{x}(\xi)$$

Thus

$$\frac{1}{2}\frac{D}{Dt}(ds^2) = (ds)\frac{D}{Dt}(ds) = \frac{\partial v_i}{\partial x_j} dx_j\, dx_i = (e_{ij} + \Omega_{ij})\, dx_j\, dx_i = e_{ij}\, dx_j\, dx_i$$

by symmetry, or

$$\frac{1}{(ds)}\frac{D}{Dt}(ds) = e_{ij}\,\frac{dx_i}{ds}\frac{dx_j}{ds} \ .$$

Now dx_i/ds is the *ith* component of a unit vector in the direction of the segment *PQ*, so that this equation says that the rate of change of the length of the segment as a fraction of its length is related to its direction through the deformation tensor.

In particular, if *PQ* is parallel to the coordinate axis *01* we have $d\mathbf{x}/ds = \mathbf{e}_{(1)}$ and

$$\frac{1}{dx_1}\frac{D}{Dt}(dx_1) = e_{11} \ \text{ in direction of } 01$$

Thus e_{11} is the rate of longitudinal strain of an element parallel to the 01 axis. Similar interpretations apply to e_{22} and e_{33}.

Now lets examine the angle between two line segments during the motion. Consider the segment, *PQ* and *PR* where *PQ* is the segment $\xi + d\xi$ as before and PR is the segment $\xi + d\xi'$. If θ is the angle between them and ds' is the length of *PR*, we have from the scalar product,

$$ds\, ds' \cos\theta = dx_i\, dx'_i$$

Differentiating with respect to time we have

$$\frac{D}{Dt}\left[ds\, ds' \cos\theta\right] = dv_i\, dx'_i + dx_i\, dv'_i$$

$$= \frac{\partial v_i}{\partial x_j} dx_j\, dx'_i + dx_i \frac{\partial v_i}{\partial x_j} dx'_j$$

since $dv'_i = (\partial v_i / \partial x_j) dx'_j$. The *i* and *j* are dummy suffixes so we may interchange them in the first term on the right, then performing differentiation we have after dividing by *ds ds'*

$$\cos\theta\left\{\frac{1}{ds}\frac{D}{Dt}ds+\frac{1}{ds'}\frac{D}{Dt}ds'\right\}-\sin\theta\frac{D\theta}{Dt}$$
$$=\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}\right)\frac{dx_j}{ds}\frac{dx'_i}{ds'}$$
$$=2e_{ij}\frac{dx'_i}{ds'}\frac{dx_j}{ds}$$

Now suppose that *dx'* is parallel to the axis *01* and *dx* to the axis *02*, so that $(dx'_i/ds')=\delta_{i1}$ and $(dx_j/ds)=\delta_{j2}$ and $\theta_{12}=\pi/2$. Then

$$-\frac{d\theta_{12}}{dt}=2e_{12}\,.$$

Thus e_{12} is to be interpreted as one-half the rate of decrease of the angle between two segments originally parallel to the *01* and *02* axes respectively. Similar interpretations are appropriate to e_{23} and e_{31}.

The fact that the rate of deformation tensor is linear in the velocity field has an important consequence. Since we may superimpose two velocity fields to form a third, it follows that the deformation tensor of this is the sum, of the deformation tensors of the fields from which it was superimposed. If $v_i= \lambda(i)\, x_i$ (no summation on *i*) , we have a deformation which is the superposition of three stretching parallel to the three axis. However, if $v_1=f(x_2)$, $v_2 = 0$, $v_3 = 0$ so that only nonzero component of the deformation tensor is $e_{12} = ½\, f'(x_2)$, the motion is one of pure shear in which elements parallel to the coordinate axis is not stretched at all. Note however that in pure stretching an element not parallel or perpendicular to the direction of stretching will suffer rotation. Likewise in pure shear an element not normal to or in the plane of shear will suffer stretching.

Principal axis of deformation

The rate of deformation tensor is a symmetric tensor and the principal axis of deformation can be found. They correspond to the eigenvalues of the matrix and the eigenvalues are the principal rates of strain. A set of particles that is originally on the surface of a sphere will be deformed to an ellipsoid whose axes are coincident with the principal axis.

Vorticity, vortex lines, and tubes

We have frequently reminders of rotating bodies of fluid such as tropical storms, hurricanes, tornadoes, dust devils, whirlpools, eddies in the flow behind objects, turbulence, and the vortex in draining bathtubs. The kinematics of these fluid motions is described by the vorticity.

The antisymmetric part of the rate of strain tensor Ω_{ij} represents the local rotation, ω_k. Recall

$$\omega_i = -\frac{1}{2}\varepsilon_{ijk}\,\Omega_{jk}$$
$$\begin{aligned}\omega &= -\mathrm{vec}\Omega \\ &= \frac{1}{2}\nabla\times\mathbf{v}\end{aligned}$$

The curl of velocity is known as the *vorticity*,

$$\mathbf{w} = \nabla\times\mathbf{v}$$

Thus the vorticity and the antisymmetric part of the rate of strain tensor is a measure of the rotation of the velocity field. An irrotational flow field is one in which the vorticity vanishes everywhere. The field lines of the vorticity field are called *vortex lines* and the surface generated by the vortex lines through a closed curve *C* is a *vortex tube*. The strength of a vortex tube is defined as the surface integral of the normal component. It is equal to the circulation around the closed curve *C* that bounds the cross-section *S* by Stokes' theorem.

$$\begin{aligned}\iint_S \mathbf{w}\bullet\mathbf{n}\,dS &= \iint_S (\nabla\times\mathbf{v})\bullet\mathbf{n}\,dS \\ &= \oint \mathbf{v}\bullet\mathbf{t}\,ds \\ &= \Gamma\end{aligned}$$

We observe that the strength of a vortex tube at any cross-section is the same, as **w** is a solenoidal vector. The surface integral of the normal component of a solenoidal vector vanishes over any closed surface. The surface integral on the surface of a vortex tube is zero because the sides are tangent to the vorticity vector field. Thus the surface integral across any cross-section must be equal in magnitude. The magnitude of the vorticity field can be visualized from the relative width of the vortex tubes in the same manner that the magnitude of the velocity field can be visualized by the width of the stream tubes.

Because the strength of the tube does not vary with position along the tube, it follows that the vortex tubes are either closed, go to infinity or end on solid boundaries of rotating objects. In a real fluid satisfying the no-slip boundary condition, vortex lines must be tangential to the surface of a body at rest, except at isolated points of attachment and separation, because the normal component of vorticity vanishes on the stationary solid.

When the vortex tube is immediately surrounded by irrotational fluid, it will be referred to as a *vortex filament*. A vortex filament is often just called a vortex, but we shell use this term to denote any finite volume of vorticity immersed in irrotational fluid. Of course, the vortex filament and the vortex require the fluid to be ideal (zero viscosity) to make strict sense, because viscosity diffuses vorticity, but they are useful approximations for real fluids of small viscosity.

Helmholtz gave three laws of vortex motion in 1858. For the motion of an ideal (zero viscosity) barotropic (density is a single valued function of pressure) fluid under the action of conservative external body forces (gradient of a scalar), they can be expressed as follows:

I. Fluid particles originally free of vorticity remain free of vorticity.

II. Fluid particles on a vortex line at any instant will be on a vortex line at subsequent times. Alternatively, it can be said that vortex lines and tubes move with the fluid.

III. The strength of a vortex tube does not vary with time during the motion of the fluid.

The equations for the dynamics of vorticity will be developed later.

P. G. Saffman, *Vortex Dynamics*, Cambridge University Press, 1992.
C. Truesdell, *The Kinematics of Vorticity*, Indiana University Press, 1954.

Assignment **4.1** Plot the streamlines, particle paths, and streaklines of the flow field described in Sec. 4.13. Find the CHBE 501 web page. Download the files in CENG501/Problems/lines. Execute *lines* with MATLAB.

Assignment **4.2** Execute the program *deform* in CHBE 501/Problems/deform and print the figures. It computes the particle paths for a patch of particles deforming in Couette flow, stagnation flow, and that of Sec 4.13. Look at the respective subroutine to determine the equations of the flow field. For each of these flow fields calculate:

a) divergence
b) curl
c) rate of deformation tensor
d) antisymmetric tensor
e) Which fields are solenoidal or irrotational?