Chapter 3 - Cartesian Vectors and Tensors: Their Calculus

Reading assignment: Chapter 3 of Aris

Tensor functions of time-like variable

In the last chapter, vectors and tensors were defined as quantities with components that transform in a certain way with rotation of coordinates. Suppose now that these quantities are a function of time. The derivatives of these quantities with time will transform in the same way and thus are tensors of the same order. The most important derivatives are the velocity and acceleration.

$$\mathbf{v}(t) = \dot{\mathbf{x}}(t), \ v_i = \frac{dx_i}{dt}$$

$$\mathbf{a}(t) = \ddot{\mathbf{x}}(t), \ a_i = \frac{d^2 x_i}{dt^2}$$

The differentiation of products of tensors proceeds according to the usual rules of differentiation of products. In particular,

$$\frac{d}{dt}\left(\mathbf{a} \bullet \mathbf{b}\right) = \frac{d\mathbf{a}}{dt} \bullet \mathbf{b} + \mathbf{a} \bullet \frac{d\mathbf{b}}{dt}$$

$$\frac{d}{dt}\left(\mathbf{a} \times \mathbf{b}\right) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$$

Curves in space

The trajectory of a particle moving is space defines a curve that can be defined with time as parameter along the curve. A curve in space is also defined by the intersection of two surfaces, but points along the curve are not associated with time. We will show that a natural parameter for both curves is the distance along the curve.

The variable position vector **x**(*t*) describes the motion of a particle. For a finite interval of *t*, say $a \le t \le b$, we ca plot the position as a curve in space. If the curve does not cross itself (i.e., if $\mathbf{x}(t) \ne \mathbf{x}(t')$, $a \le t < t' \le b$) it is called *simple*; if $\mathbf{x}(a) = \mathbf{x}(b)$ the curve is *closed*. The variable *t* is now just a parameter along the curve that may be thought of as the time in motion of the particle. If *t* and *t*' are the parameters of two points, the cord joining them is the vector $\mathbf{x}(t') - \mathbf{x}(t)$. As $t \to t'$ this vector approaches $\left(t'-t\right)\ \dot{\mathbf{x}}(t)$ and so in the limit is proportional to $\dot{\mathbf{x}}(t)$. However the limit of the cord is the tangent so that $\dot{\mathbf{x}}(t)$ is in the direction of the tangent. If $v^2 = \dot{\mathbf{x}}(t) \bullet \dot{\mathbf{x}}(t)$ we can construct a *unit tangent vector* $\boldsymbol{\tau}$.

$$\boldsymbol{\tau} = \dot{\mathbf{x}}(t)/v = \mathbf{v}/v$$

Now we will parameterize a curve with distance along the curve rather than time. If **x**(*t*) and **x**(*t*+*dt*) are two very close points,

$$\mathbf{x}(t+dt) = \mathbf{x}(t) + dt\,\dot{\mathbf{x}}(t) + O(dt^2)$$

and the distance between them is

$$\begin{aligned} ds^2 &= \left\{\mathbf{x}(t+dt) - \mathbf{x}(t)\right\} \bullet \left\{\mathbf{x}(t+dt) - \mathbf{x}(t)\right\} \\ &= \dot{\mathbf{x}}(t) \bullet \dot{\mathbf{x}}(t)\, dt^2 + O(dt^3) \end{aligned}$$

The arc length from any given point *t*=*a* is therefore

$$s(t) = \int_a^t \left[\dot{\mathbf{x}}(t') \bullet \dot{\mathbf{x}}(t')\right]^{1/2} dt'.$$

s is the natural parameter to use on the curve, and we observe that

$$\frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \frac{\dot{\mathbf{x}}(t)}{v} = \boldsymbol{\tau}$$

A curve for which a length can be so calculated is called *rectifiable*. From this point on we will regard *s* as the parameter, identifying *t* with *s* and letting the dot denote differentiation with respect to *s*. Thus

$$\boldsymbol{\tau}(s) = \dot{\mathbf{x}}(s)$$

is the unit tangent vector.

Let **x**(*s*), **x**(*s*+*ds*), and **x**(*s*-*ds*) be three nearby points on the curve. A plane that passes through these three points is defined by the linear combinations of the cord vectors joining the points. This plane containing the points must also contain the vectors

$$\frac{\mathbf{x}(s+ds)-\mathbf{x}(s)}{ds}=\dot{\mathbf{x}}(s)+O(ds)$$

and

$$\frac{\mathbf{x}(s+ds)\text{ - }2\,\mathbf{x}(s)+\mathbf{x}(s\text{ - }ds)}{ds^2}=\ddot{\mathbf{x}}(s)+O(ds^2)$$

Thus, in the limit when the points are coincident, the plane reaches a limiting position defined by the first two derivatives $\dot{\mathbf{x}}(s)$ and $\ddot{\mathbf{x}}(s)$. This limiting plane is called the *osculating plane* and the curve appears to lie in this plane in the intermediate neighborhood of the point. To prove this statement: (1) A plane is defined by the two vectors, $\dot{\mathbf{x}}(s)$ and $\ddot{\mathbf{x}}(s)$, if they are not co-linear. (2) The coordinates of the three points on the curve in the previous two equations are a linear combination of $\mathbf{x}(s), \dot{\mathbf{x}}(s)$ and $\ddot{\mathbf{x}}(s)$, thus they line in the plane.

Now $\dot{\mathbf{x}}=\boldsymbol{\tau}$ so $\ddot{\mathbf{x}}=\dot{\boldsymbol{\tau}}$ and since $\boldsymbol{\tau}\bullet\boldsymbol{\tau} = 1$,

$$\frac{d\left(\boldsymbol{\tau}\bullet\boldsymbol{\tau}\right)}{ds}=0=\dot{\boldsymbol{\tau}}\bullet\boldsymbol{\tau}+\boldsymbol{\tau}\bullet\dot{\boldsymbol{\tau}}=2\,\boldsymbol{\tau}\bullet\dot{\boldsymbol{\tau}}$$

$$\boldsymbol{\tau}\bullet\dot{\boldsymbol{\tau}}=0$$

so that the vector $\dot{\boldsymbol{\tau}}$ is at right angles to the tangent. Let $1/\rho$ denote the magnitude of $\dot{\boldsymbol{\tau}}$.

$$\dot{\boldsymbol{\tau}}\bullet\dot{\boldsymbol{\tau}}=\frac{1}{\rho^2}$$

and

$$\mathbf{v}=\rho\,\dot{\boldsymbol{\tau}}$$

Then $\mathbf{v}$ is a unit normal and defines the direction of the so-called principle normal to the curve.

To interpret ρ, we observe that the small angle $d\theta$ between the tangents at s and $s+ds$ is given by

$$\cos d\theta=\boldsymbol{\tau}(s)\bullet\boldsymbol{\tau}(s+ds)$$

$$1-\frac{1}{2}d\theta^2+...=\boldsymbol{\tau}\bullet\boldsymbol{\tau}+\boldsymbol{\tau}\bullet\dot{\boldsymbol{\tau}}\,ds+\frac{1}{2}\boldsymbol{\tau}\bullet\ddot{\boldsymbol{\tau}}\,ds^2+...$$

$$=1-\frac{1}{2}\dot{\boldsymbol{\tau}}\bullet\dot{\boldsymbol{\tau}}\,ds^2+...$$

since $\boldsymbol{\tau}\bullet\dot{\boldsymbol{\tau}}=0$ and so $\boldsymbol{\tau}\bullet\ddot{\boldsymbol{\tau}}+\dot{\boldsymbol{\tau}}\bullet\dot{\boldsymbol{\tau}}=0$. Thus,

$$\rho = \left| \frac{ds}{d\theta} \right|$$

is the reciprocal of the rate of change of the angle of the tangent with arc length, i.e., ρ is the radius of curvature. Its reciprocal 1/ρ is the curvature, $\kappa \equiv \left| d\theta / ds \right| = 1/\rho$.

A second normal to the curve may be taken to form a right-hand system with τ and ν. This is called the unit binormal,

$$\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu}$$

Line integrals

If *F*(**x**) is a function of position and *C* is a curve composed of connected arcs of simple curves, **x** = **x**(*t*), *a* ≤ *t* ≤ *b* or **x** = **x**(*s*), *a* ≤ *s* ≤ *b*, we can define the integral of *F* along *C* as

$$\int_C F(\mathbf{x}) dt = \int_a^b F\left[\mathbf{x}(t)\right] dt$$

or

$$\int_C F(\mathbf{x}) ds = \int_a^b F\left[\mathbf{x}(t)\right] \left\{ \dot{\mathbf{x}}(t) \bullet \dot{\mathbf{x}}(t) \right\}^{1/2} dt$$

Henceforth, we will assume that the curve has been parameterized with respect to distance along the curve, *s*.

The integral is from *a* to *b*. If the integral is in the opposite direction with opposite limits, then the integral will have the same magnitude but opposite sign. If **x**(*a*) = **x**(*b*), the curve *C* is closed and the integral is sometimes written

$$\oint_C F[\mathbf{x}(s)] ds$$

If the integral around any simple closed curve vanishes, then the value of the integral from any pair of points *a* and *b* is independent of path. To see this we take any two paths between *a* and *b*, say C_1 and C_2, and denote by *C* the closed path formed by following C_1 from *a* to *b* and C_2 back from *b* to *a*.

$$\begin{aligned} \oint_C F\, ds &= \left[\left[\int_a^b F\, ds \right]\right]_{C_1} + \left[\left[\int_b^a F\, ds \right]\right]_{C_2} \\ &= \left[\left[\int_a^b F\, ds \right]\right]_{C_1} - \left[\left[\int_a^b F\, ds \right]\right]_{C_2} \\ &= 0 \end{aligned}$$

If **a**(**x**) is any vector function of position, **a**•τ is the projection of **a** tangent to the curve. The integral of **a**•τ around a simple closed curve *C* is called the *circulation* of **a** around *C*.

$$\oint_C \mathbf{a} \bullet \boldsymbol{\tau}\, ds = \oint_C a_i \left[x_1(s), x_2(s), x_3(s) \right] \tau_i \, ds$$

We will show later that a vector whose circulation around any simple closed curve vanishes is the gradient of a scalar.

Surface integrals

Many types of surfaces and considered in transport phenomena. Most often the surfaces are the boundaries of volumetric region of space where boundary conditions are specified. The surfaces could also be internal boundaries where the material properties change between two media. Finally the surface itself may the subject of interest, e.g. the statics and dynamics of soap films.

A proper mathematical treatment of surfaces requires some definitions. A *closed surface* is one which lies within a bounded region of space and has an inside and outside. If the normal to the surface varies continuously over a part of the surface, that part is called *smooth*. The surface may be made up of a number of subregions, which are smooth and are called *piece-wise smooth*. A closed curve on a surface, which can be continuously shrunk to a point, is called *reducible*. If all closed curves on a surface are reducible, the surface is called *simply connected*. The sphere is simply connected but a torus is not.

If a surface is not closed, it normally has a space curve as its boundary, as for example a hemisphere with the equator as boundary. It has two sides if it is impossible to go from a point on one side to the other along a continuous curve that does not cross the boundary curve. The surface is sometimes called the *cap* of the space curve.

If *S* is a piece-wise smooth surface with two sides in three-dimensional space, we can divide it up into a large number of small surface regions such that the dimensions of the regions go to zero as the number of regions go to infinity. If the regions fill the surface and are not overlapping, then sum of the areas of the regions is equal to the area of the surface. If the function, *F* is defined on the surface, it can be evaluated for some point of each subregion of the surface and the sum $\Sigma\, F\, \Delta S$ computed. The limit as the number of regions go to infinity and the dimensions of the regions go to zero is *surface integral* of *F* over *S*.

$$\lim \sum F\, \Delta S = \iint_S F\, dS$$

The traditional symbol of the double integral is retained because if the surface is a plane or the surface is projected on to a plane, then Cartesian coordinates can be defined such that the surface integral is a double integral of the two coordinates in the plane. Also, two surface coordinates can define a surface and the double integration is over the surface integrals.

In transport phenomena the surface integral usually represents the flow or flux of a quantity across the surface and the function *F* is the normal component of a vector or the contracted product of a tensor with the unit normal vector. Thus one needs to know the direction of the normal in addition to the differential area to calculate the surface integral. Consider the case of a surface defined as a function of two surface coordinates.

$$\mathbf{x}(u_1,u_2)=\mathbf{f}(u_1,u_2) \;\text{ on }\; S$$

$$\mathbf{n}\,dS=\left(\frac{\partial\mathbf{f}}{\partial u_1}\times\frac{\partial\mathbf{f}}{\partial u_2}\right)du_1\,du_2$$

To see how we arrive at this result, recall the partial derivatives of the coordinates of a curve with respect to a parameter is a vector that is tangent to the curve. The magnitude is

$$\begin{aligned}\left|\frac{\partial\mathbf{f}}{\partial u_i}\right| &= \left(\frac{\partial\mathbf{f}}{\partial u_i}\bullet\frac{\partial\mathbf{f}}{\partial u_i}\right)^{1/2}\\ &= \left[\left(\frac{\partial f_1}{\partial u_i}\right)^2+\left(\frac{\partial f_2}{\partial u_i}\right)^2+\left(\frac{\partial f_3}{\partial u_i}\right)^2\right]^{1/2}\\ &= \left|\left(\frac{ds}{du_i}\right)_{u_j}\right|\end{aligned}$$

The vector product has a magnitude equal to the product of the magnitudes and the sine of the angle between the vectors. This gives us the area of a parallelogram corresponding to the area of the differential region.

$$dS=\left|\left(\frac{ds}{du_1}\right)_{u2}\right|\;\left|\left(\frac{ds}{du_2}\right)_{u1}\right|\sin\theta\,du_1\,du_2$$

The two tangent vectors in the direction of the surface coordinates lie in the tangent plane of the surface. Thus the direction of the vector product is perpendicular to the surface. Inward or outward direction for the normal has not yet been specified and will be determined by the sign.

Volume integrals

The volume integral of a function *F* over a volumetric region of space *V* is the limit of the sum of the products of the volume of small volumetric subregions of *V* and the function *F* evaluated somewhere within each subregion.

$$\iiint_V F(\mathbf{x})\, dV = \lim \sum F\, \Delta V$$

Change of variables with multiple integrals

In Cartesian coordinates the elements of volume dV is simply the volume of a rectangular parallelepiped of sides dx_1, dx_2, dx_3 and so

$$dV = dx_1\, dx_2\, dx_3$$

Suppose, however, that it is convenient to describe the position by some other coordinates, say ξ_1, ξ_2, ξ_3. We may ask what volume is to be associated with the three small changes $d\xi_1, d\xi_2, d\xi_3$.

The change of coordinates must be given by specifying the Cartesian point $\mathbf{x}$ that is to correspond to a given set ξ_1, ξ_2, ξ_3, by

$$x_i = x_i(\xi_1,\ \xi_2,\ \xi_3)\ .$$

Then by partial differentiation the small differences corresponding to a change $d\xi_i$ are

$$dx_i = \frac{\partial x_i}{\partial \xi_j} d\xi_j$$

Let $d\mathbf{x}^{(j)}$ be the vectors with the components $(\partial x_i/\partial \xi_j)\, d\xi_j$ for j = 1, 2, and 3. Then the volume element is

$$\begin{aligned} dV &= d\mathbf{x}^{(1)} \bullet \left(d\mathbf{x}^{(2)} \times d\mathbf{x}^{(3)} \right) \\ &= \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} d\xi_1 \frac{\partial x_j}{\partial \xi_2} d\xi_2 \frac{\partial x_k}{\partial \xi_3} d\xi_3 \\ &= J\, d\xi_1\, d\xi_2\, d\xi_3 \end{aligned}$$

where

$$J = \frac{\partial\left(x_1, x_2, x_3\right)}{\partial\left(\xi_1, \xi_2, \xi_3\right)} = \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3}$$

is called the *Jacobian* of the transformation of variables

Vector fields

When the components of a vector or tensor depend on the coordinates we speak of a vector or tensor field. The flow of a fluid is a perfect realization of a vector field for at each point in the region of flow we have a velocity vector $\mathbf{v}(\mathbf{x})$.

If the flow is unsteady then the velocity depends on the time as well as position, **v**=**v**(**x**,*t*).

Associated with any vector field **a**(**x**) are its *trajectories*, which is the name given to the family of curves everywhere tangent to the local vector **a**. They are solutions of the simultaneous equations

$$\frac{d\mathbf{x}}{ds} = \mathbf{a}(\mathbf{x}); \text{ that is } \frac{dx_i}{ds} = a_i(x_1, x_2, x_3)\,.$$

Where *s* is a parameter along the tajectory. (It will be arc length if **a** is always a unit vector.) Streamlines of a steady flow are a realization of these trajectories. For a time dependent vector field the trajectories will also be time dependent. If *C* is any closed curve in the vector field and we take the trajectories through all points of *C*, they describe a surface known as the *vector tube* of the field. For flow fields, it is called a *stream tube*.

The vector operator ∇ -gradient of a scalar

The symbol ∇ (enunciated as "*del*") is used for the symbolic vector operator whose i^{th} component is $\partial/\partial x_i$. Thus if ∇ operates on a scalar function of position $\varphi(\mathbf{x})$ it produces a vector $\nabla\varphi$ with components $\partial\varphi/\partial x_i$.

$$\text{grad}\,\varphi = \nabla\varphi = \varphi_{,i} = \mathbf{e}_{(1)}\frac{\partial\varphi}{\partial x_1} + \mathbf{e}_{(2)}\frac{\partial\varphi}{\partial x_2} + \mathbf{e}_{(3)}\frac{\partial\varphi}{\partial x_3}$$

We should establish that $\nabla\varphi$ is indeed a vector. In the coordinate frame 0$\underline{123}$ the vector $\underline{\nabla\varphi}$ will have components $\partial\varphi/\partial\underline{x}_l$. However,

$$\frac{\partial\varphi}{\partial\overline{x}_j} = \frac{\partial\varphi}{\partial x_i}\frac{\partial x_i}{\partial\overline{x}_j} = l_{ij}\frac{\partial\varphi}{\partial x_i}$$

since $\overline{a}_j = l_{ij}\,a_i$ so $\nabla\varphi$ is a vector.

The suffix notation ,*i* for the partial derivative with respect to x_i is a very convenient one and will be taken over for the generalization of this operation that must be made for non-Cartesian frame of reference. The notation "grad" for ∇ is often used and referred to as the gradient operator. Thus grad φ is the vector which is the gradient of the scalar. The gradient operator can also operate on higher order tensors and the operation raises the order by one. Thus the gradient of a vector **a** is grad **a,** ∇**a**, or in component notation $a_{i,j}$. ∇ is sometimes written $\partial/\partial\mathbf{x}$ and can be expanded as the vector operator

$$\nabla \equiv \mathbf{e}_{(i)}\frac{\partial}{\partial x_i} = \mathbf{e}_{(1)}\frac{\partial}{\partial x_1} + \mathbf{e}_{(2)}\frac{\partial}{\partial x_2} + \mathbf{e}_{(3)}\frac{\partial}{\partial x_3}$$

The gradient of a scalar gives the differential of the scalar in the direction of a differential displacement vector $d\mathbf{x}$. To see this the differential of $\varphi(\mathbf{x}) = \varphi(x_1,x_2,x_3)$ is the total derivative

$$
\begin{aligned}
d\varphi &= \frac{\partial \varphi}{\partial x_i} dx_i \\
&= \frac{\partial \varphi}{\partial x_1} dx_1 \; + \frac{\partial \varphi}{\partial x_2} dx_2 \quad \frac{\partial \varphi}{\partial x_3} dx_3 \\
&= \left(\mathbf{e}_{(1)} \frac{\partial \varphi}{\partial x_1} + \mathbf{e}_{(2)} \frac{\partial \varphi}{\partial x_2} + \mathbf{e}_{(3)} \frac{\partial \varphi}{\partial x_3} \right) \bullet \left(\mathbf{e}_{(1)} \, dx_1 + \mathbf{e}_{(2)} \, dx_2 + \mathbf{e}_{(3)} \, dx_3 \right) \\
&= \nabla \varphi \bullet d\mathbf{x}
\end{aligned}
$$

The unit vector in the direction of $d\mathbf{x}$ is $\mathbf{u} = d\mathbf{x}/ds$. The derivative of φ in the direction of $\mathbf{u}$ is

$$
\frac{d\varphi}{ds} = \nabla \varphi \bullet \mathbf{u} = \left| \nabla \varphi \right| \cos \theta
$$

If $\varphi(\mathbf{x}) = c$ is a surface, then $\nabla \varphi$ is normal to the surface. To prove this, let $d\mathbf{x}$ be a differential distance on the surface. The differential of φ along $d\mathbf{x}$ is zero for any $d\mathbf{x}$ on the surface. This implies that the scalar product of $\nabla \varphi$ with any vector on the surface is zero or that $\nabla \varphi$ has zero component or projection on the tangent plane and thus $\nabla \varphi$ is normal to the surface. Also, since $\nabla \varphi$ is normal to the surface, the derivative of φ is a maximum in the direction normal to the surface.

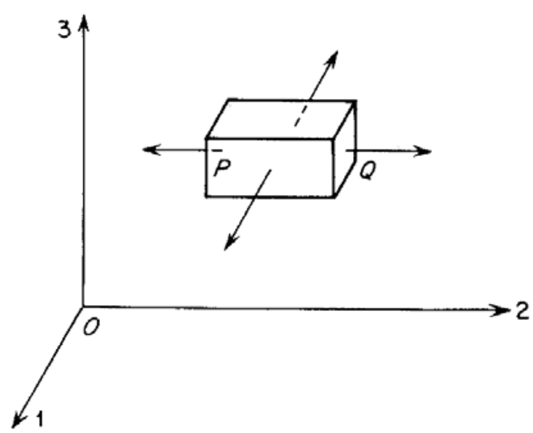
The divergence of a vector field

The symbolic scalar or dot product of the operator ∇ and a vector is called the *divergence* of the vector field. Thus for any differentiable vector field $\mathbf{a}(\mathbf{x})$ we write

$$
\operatorname{div} \mathbf{a} = \nabla \bullet \mathbf{a} = a_{i,i} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}
$$

The divergence is a scalar because it is the scalar product and because it is the contraction of the second order tensor $a_{i,j}$.

We will now demonstrate why the $\nabla \bullet$ operation on a vector field is called the divergence. Suppose that an elementary parallelepiped is set up with one corner P at x_1, x_2, x_3 and the diagonally opposite one Q at $x_1+dx_1,\ x_2+dx_2,\ x_3+dx_3$ as shown in Fig. 3.7. The outward unit normal to the face



through *Q* which is perpendicular to *01* is $\mathbf{e}_{(1)}$, whereas the outward normal to the parallel face through *P* is $-\mathbf{e}_{(1)}$. Suppose $\mathbf{a}(\mathbf{x})$ is a differentiable flux vector field. We are going to compute the net flux of **a** across the bounding surfaces of the parallelepiped. The value of the normal component of **a** at some point on the two faces perpendicular to the *01* direction are

$$a_1(x_1,\ \bar{x}_2,\ \bar{x}_3) \text{ and } a_1(x_1+dx_1,\ \bar{x}_2,\ \bar{x}_3)$$
$$\text{where}$$
$$\mathrm{x}_2 \le \bar{x}_2 \le x_2+dx_2 \text{ and } \mathrm{x}_3 \le \bar{x}_3 \le x_3+dx_3$$

Thus if **n** denotes the outward normal and *dS* is the area $dx_2\ dx_3$ of these two faces, we have a contribution from them to the surface integral $\oiint_S \mathbf{a}\bullet\mathbf{n}\, dS$ of

$$\iint_{01\,\text{faces}} \mathbf{a}\bullet\mathbf{n}\, dS = \left[a_1(x_1+dx_1,\ \bar{x}_2,\ \bar{x}_3) - a_1(x_1,\ \bar{x}_2,\ \bar{x}_3)\right] dx_2\, dx_3$$
$$= \frac{\partial a_1}{\partial x_1} dx_1\, dx_2\, dx_3 + O(dx^4)$$

where $O(dx^4)$ denotes terms proportional to fourth power of *dx*. Similar terms with $\partial a_2/\partial x_2$ and $\partial a_3/\partial x_3$ will be given by contributions of the other faces so that for the whole parallelepiped whose volume $dV = dx_1\ dx_2\ dx_3$ we have

$$\frac{1}{dV}\iint_S \mathbf{a}\bullet\mathbf{n}\, dS = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} + O(dx)$$

If we let the volume shrink to zero we have

$$\lim_{dV\to 0} \frac{1}{dV}\iint_S \mathbf{a}\bullet\mathbf{n}\, dS = \nabla\bullet\mathbf{a}$$

If **a** is a flux, then the surface integral is the net flux of **a** out of the volume. In particular, let **a** be the fluid velocity, which can be thought as a volumetric flux. Then the divergence of velocity is the volumetric expansion per unit volume. A vector field with identically zero divergence is called *solenoidal*. An incompressible fluid has a solenoidal velocity field. If the flux field of a certain property is solenoidal there is no generation of that property within the field, for all that flows into an infinitesimal element flows out again.

If **a** is the gradient of a scalar function $\nabla\varphi$, its divergence is called the *Laplacian* of φ.

$$\nabla^2\varphi = \nabla\bullet\left(\nabla\varphi\right) = \varphi_{,ii} = \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2}$$

A function that satisfies Laplace's equation $\nabla^2\varphi = 0$ is called a *potential function* or a *harmonic function*. An irrotational, incompressible flow field has a velocity that is the gradient of a flow potential. Also, the steady-state temperature field in a homogeneous solid and the steady state pressure distribution of a single fluid phase flowing in porous media are solutions of Laplace's equation. In two dimensions the solutions of Laplace's equation can be found through the use of complex variables.

If **A** is a tensor, the notation div **A** or ∇•**A** is sometimes used for the vector $A_{ij,i}$. The index notation is preferred for tensors.

The curl of a vector field

The symbolic vector product or cross product of the vector operator ∇ and a vector field **a**(**x**) is called the *curl* of the vector field. It is the vector

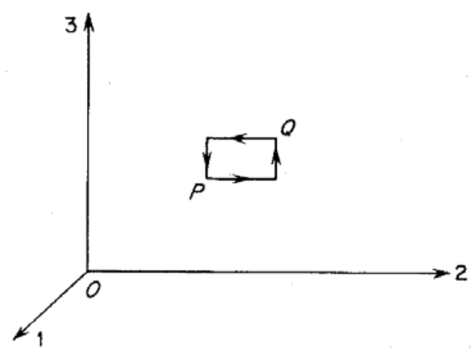
$$\nabla \times \mathbf{a} = \operatorname{curl} \mathbf{a} = \varepsilon_{ijk}\, a_{k,j}\, \mathbf{e}_{(i)}$$

That this definition is a combination of the previously definitions for the ∇ operator and the cross product can be seen be carrying out the operations.

$$\begin{aligned}\nabla \times \mathbf{a} &= \left(\mathbf{e}_{(1)}\frac{\partial}{\partial x_1} + \mathbf{e}_{(2)}\frac{\partial}{\partial x_2} + \mathbf{e}_{(3)}\frac{\partial}{\partial x_3}\right)\times\left(\mathbf{e}_{(1)}\,a_1 + \mathbf{e}_{(2)}\,a_2 + \mathbf{e}_{(3)}\,a_3\right)\\ &= \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\mathbf{e}_{(1)} + \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right)\mathbf{e}_{(2)} + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathbf{e}_{(3)}\end{aligned}$$

also

$$\begin{aligned}\nabla \times \mathbf{a} &= \varepsilon_{ijk}\frac{\partial a_j}{\partial x_i}\mathbf{e}_{(k)} = \varepsilon_{ijk}\, a_{j,i}\,\mathbf{e}_{(k)} = \varepsilon_{kji}\, a_{j,k}\,\mathbf{e}_{(i)} = \varepsilon_{jki}\, a_{k,j}\,\mathbf{e}_{(i)}\\ &= \varepsilon_{ijk}\, a_{k,j}\,\mathbf{e}_{(i)} \quad Q.E.D.\end{aligned}$$



The connection between the curl of a vector field and the rotation of the vector field (it is called rot **a** in some older texts) can be illustrated by calculating the circulation of the vector field around a closed curve. Consider a elementary rectangle in the *023* plane normal to *01* with one corner *P* at (x_1, x_2, x_3) and the diagonally opposite one *Q* at (x_1, x_2+dx_2, x_3+dx_3) as shown in Fig. 3.8. We wish to calculate the line integral or circulation around this elementary closed curve of **a•t** *ds*, where **t** is the unit tangent to the curve. Now the line through *P* parallel to *03* has tangent –$\mathbf{e}_{(3)}$ and the parallel side through *Q* has tangent $\mathbf{e}_{(3)}$, and each is of length dx_3. Accordingly, they contribute to **a•t** *ds* an amount

$$\left[a_3(x_1,\ \ x_2+dx_2,\ \ \bar{x}_3)-a_3(x_1,\ \ x_2,\ \ \bar{x}_3)\right]dx_3=\frac{\partial a_3}{\partial x_2}dx_2\,dx_3+O(dx^3)$$

Similarly, from the other two sides, there is a contribution,

$$-\frac{\partial a_2}{\partial x_3}dx_3\,dx_2+O(dx^3)$$

Thus writing $dA = dx_2\, dx_3$, we have

$$\frac{1}{dA}\oint_{023}\mathbf{a}\bullet\mathbf{t}\quad ds=\left(\frac{\partial a_3}{\partial x_2}-\frac{\partial a_2}{\partial x_3}\right)+O(dx)$$

and in the limit

$$\lim_{dA\to 0}\ \frac{1}{dA}\oint_{023}\mathbf{a}\bullet\mathbf{t}\quad ds=\left(\frac{\partial a_3}{\partial x_2}-\frac{\partial a_2}{\partial x_3}\right)=\left(\nabla\times\mathbf{a}\right)_1=\left(\nabla\times\mathbf{a}\right)\bullet\mathbf{n}$$

The suffix *023* has been put on the integral sign to show that the line integral in a *023* plane, and the last equation shows that the circulation in the *O23* plane is equal to the component of the curl in the *O1* direction. This correspondence between the curl and circulation gives physical meaning the curl of a vector field. It is a measure of the circulation or rotation of the motion. There is a direction associated with circulation, rotation, and curl. If the circulation around a closed curve is in the direction of the closed fingers of the right hand, then the curl is in the direction of the thumb.

A vector field **a** for which $\nabla\times\mathbf{a} = 0$ is called *irrotational* because the circulation about any closed curve vanishes.

There are several important identities involving the curl operator.

$$\nabla\bullet(\nabla\times\mathbf{a})=0$$
$$\nabla\times(\nabla\varphi)=0$$
$$\nabla\times(\varphi\quad\mathbf{a})=(\nabla\varphi)\times\mathbf{a}+\varphi\quad\nabla\times\mathbf{a}$$
$$\nabla\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}(\nabla\bullet\mathbf{b})-\mathbf{b}(\nabla\bullet\mathbf{a})+(\mathbf{b}\bullet\nabla)\mathbf{a}-(\mathbf{a}\bullet\nabla)\mathbf{b}$$
$$\nabla\times(\nabla\times\mathbf{a})=\nabla(\nabla\bullet\mathbf{a})-\nabla^2\mathbf{a}$$

The first of these states that if a vector field **b** is the curl of a vector, i.e., $\mathbf{b} = \nabla\times\mathbf{a}$ then the vector field **b** is solenoidal. The second states that if a vector field **b** is equal to the gradient of a scalar, i.e., $\mathbf{b} = \nabla\varphi$ then the vector field **b** is irrotational. The last identity has the Laplacian operator $\nabla^2 = \nabla\bullet\nabla$ operating on a vector. The result is a vector whose components are equal to the Laplacian of the components, if the coordinates are Cartesian. This may not be the case in curvilinear coordinates.

Green's theorem and some of its variants

The divergence theorem, also called the Gauss' theorem, or Green's theorem equates the volume integral of the divergence of a vector field **a**(**x**) to the surface integral of the normal component of the vector.

$$\iiint_V \nabla \bullet \mathbf{a}\, dV = \oiint_S \mathbf{a} \bullet \mathbf{n}\, dS$$

During the discussion of the divergence of a vector field, we showed that the above relation holds for an infinitesimal volume. Suppose now that a macroscopic volume of space is a composite of the infinitesimal regions. The total volume integral is the sum of the infinitesimal volume integrals. However, the contribution of **a•n** *dS* from the touching faces of two adjacent elements of volume are equal in magnitude but opposite in sign since the outward normal points in opposite directions. Thus in a summation of **a•n** *dS,* the only terms that survive are those on the outer surface *S*, i.e., the surface integral is over the exterior surfaces of the macroscopic region. Q.E.D.

If **a** = $\nabla\varphi$ we have

$$\iiint_V \nabla^2 \varphi\, dV = \iiint_V \nabla \bullet \nabla \varphi\, dV = \oiint_S \nabla \varphi \bullet \mathbf{n}\, dS = \oiint_S \frac{\partial \varphi}{\partial n} dS$$

where $\partial\varphi/\partial n$ denotes the derivative in the direction of the outward normal. If the scalar φ is temperature, this equation says that at steady-state, the integral of the net sources of heat in the volume is equal to the flux across the external surfaces.

Stokes' theorem

On a surface let *C* be the curve or a finite number of curves forming the complete boundary of an area *S*. We assume that the surface is two-sided and that S can be resolved into a finite number of regular elements. Choose a positive side of *S* and let the positive direction along *C* be that in which an observer on the positive side must move along the boundary if he is to have the area *S* always on his left. At each regular point on the surface let **n** be the unit normal drawn toward the positive side. Let **a** and its first derivatives be continuous on *S*. Stokes' theorem states that the circulation around a closed curve is equal to the surface integral of the normal component of the curl.

$$\oint_C \mathbf{a} \bullet \mathbf{t}\, ds = \iint_S (\nabla \times \mathbf{a}) \bullet \mathbf{n}\, dS$$

We showed earlier the circulation around a infinitesimal, closed curve was equal to the normal component of the curl multiplied by the area of the enclosed surface. We will extend the earlier result for an infinitesimal closed curve enclosing an infinitesimal surface to a macroscopic curve and surface. The macroscopic surface will be subdivided into a composite of many infinitesimal regions where the earlier result apply. The summation of the normal component of the curl multiplied by the area of the element is equal to the surface integral of the normal component of the curl. However, the quantity $\mathbf{a} \bullet \mathbf{t}\ ds$ from the touching sides of two adjacent surface elements have equal magnitude but opposite sign since the direction of the line integrals are in opposite directions. Thus in the summation of the circulations, the only terms that survive are the contribution of the external bounding curve, i.e., the circulation is around the exterior curve *C* bounding the surface *S*. Q.E.D.

The classification and representation of vector fields

We mentioned earlier that a solenoidal vector field is one where $\nabla \bullet \mathbf{a} = 0$ everywhere and an irrotational vector field is one where $\nabla \times \mathbf{a} = 0$ everywhere. A vector field that is the gradient of a scalar $\mathbf{a} = \nabla \varphi$ is irrotational. If a vector field is both irrotational and solenoidal it is the gradient of a harmonic function, where $\nabla \bullet (\nabla \varphi) = \nabla^2 \varphi = 0$. It can be proven that if a vector field is both irrotational and solenoidal, it is uniquely determined in a volume *V* if it is specified over *S*, the surface of *V*.

There other types of named vector fields are discussed by Aris.

Irrotational vector fields

The vector field **a** is irrotational if its curl vanishes everywhere. By Stokes' theorem the circulation around any closed curve also vanishes. Also, an irrotational vector field can be expressed as the gradient of a scalar.

$$\left.\begin{array}{c} \nabla \times \mathbf{a} = 0 \\ \oint_C \mathbf{a} \bullet \mathbf{t} \; ds = 0 \\ \mathbf{a} = \nabla \varphi \end{array}\right\} \; \mathbf{a} \text{ an irrotational field}$$

The velocity field of motions where the viscous effects are insignificant compared to inertial effects and the flow is initially irrotational can be approximated as an irrotational velocity field.

Solenoidal vector fields

A solenoidal vector field is defined as one in which the divergence vanishes. This implies that the flux across a closed surface must also vanish. A vector identity states that the divergence of the curl of a vector is zero. Thus a continuously differentiable solenoidal vector field has the following three equivalent characteristics.

$$
\left.\begin{array}{c}
\nabla \bullet \mathbf{a} = 0 \\
\oiint_S \mathbf{a} \bullet \mathbf{n} \;\; dS = 0 \\
\mathbf{a} = \nabla \times \mathbf{A}
\end{array}\right\} \;\; \mathbf{a} \;\; \text{a solenoidal field}
$$

The velocity field of motions where the effects of compressibility are insignificant can be approximated as a solenoidal vector field. The surface integral of velocity vanishing over any closed surface means that the net volumetric flow across closed surfaces is zero. Incompressible flow fields can be expressed as the curl of a vector potential. Two-dimensional, incompressible flows have only one nonzero component of the vector potential and this is identified as the stream function.

Helmholtz' representation

We found that an irrotational vector is the gradient of a scalar potential and a solenoidal vector is the curl of a vector potential. Here we show that any vector field with sufficient continuity is divisible into irrotational and solenoidal parts, and so is expressible in terms of a scalar and a vector potential. The fundamental problem in the analysis of a vector field is the determination of these potentials and their expression in terms of the essential characteristics of the vector, namely divergence, curl, discontinuities, and boundary values. For when the potentials are known the vector itself can be determined by differentiation. The following analysis is taken from H. B. Phillips, *Vector Analysis*, John Wiley & Sons, 1933. The following nomenclature will differ somewhat in that the vector is expressed as the negative of the gradient of a scalar. Also, the vector field of interest will be denoted as **F**. Bold face capital letters will also be used for other vector quantities. Also the equations have the 4π factor of electromagnetism in mks units rather than the factors ε_o and $\varepsilon_o c^2$ of the SI units.

Let V be a region of space where the vector field **F** has piecewise continuous second derivatives, S_1 be surfaces of discontinuity of **F**, and S be the bounding surface of V. The Helmholtz's theorem states that **F** can be expressed in terms of the potentials.

$$\mathbf{F} = -\nabla\varphi + \nabla\times\mathbf{A}$$

where

$$\varphi(\mathbf{x}_P) = \iiint\limits_V \frac{\rho \quad dV}{r} + \iint\limits_{S_1} \frac{\sigma \quad dS}{r} - \frac{1}{4\pi} \oiint\limits_S \frac{\mathbf{n}\bullet\mathbf{F} \quad dS}{r}$$

$$\mathbf{A}(\mathbf{x}_P) = \iiint\limits_V \frac{\mathbf{I} \quad dV}{r} + \iint\limits_{S_1} \frac{\mathbf{J} \quad dS}{r} - \frac{1}{4\pi} \oiint\limits_S \frac{\mathbf{n}\times\mathbf{F} \quad dS}{r}$$

$$\nabla\bullet\mathbf{F} = -\nabla^2\varphi = 4\pi\rho$$

$$\nabla\times\mathbf{F} = \nabla\times\left(\nabla\times\mathbf{A}\right) = 4\pi\mathbf{I}$$

$$\mathbf{n}\bullet\Delta\mathbf{F} = 4\pi\sigma$$

$$\mathbf{n}\times\Delta\mathbf{F} = 4\pi\mathbf{J}$$

$$r = \left|\mathbf{x}_P - \mathbf{x}_Q\right| \text{ where } \mathbf{x}_Q \text{ is coordinate of integrand}$$

The vectors **I** and **J** are not arbitrary. They are subject to the equation of continuity ∇•**A** = 0. The effect of this condition is to make **I** and **J** behave like space and surface currents of something which is nowhere created or destroyed. In the electromagnetic field they usually represent currents of electricity. In hydrodynamic fields they represent vorticity. If **A** is everywhere solenoidal, the following three equations must then be everywhere satisfied.

$$\nabla\bullet\mathbf{I} = 0$$

$$\mathbf{n}\bullet\Delta\mathbf{I} + \nabla\bullet\mathbf{J} = 0$$

$$\mathbf{n}\times\Delta\mathbf{J}\bullet\mathbf{t} \quad ds = 0$$

Considering **I** and **J** as representing currents, the first equation expresses that the amount which flows out of a small region is equal to the amount which flows in. The second equation expresses that the total flow from a portion of a conducting surface into space and along the surface is zero. The third equation expresses that the flow from one sub-region across a curve on a conducting surface is equal to the flow into the adjacent sub-region. These three equations thus express that **I** and **J**, considered as space and surface currents, represent a flow of something which is conserved. For **I** and **J** to have this property the above discussion shows it is necessary and sufficient that **A** be everywhere solenoidal. In hydrodynamics, the field **I** corresponds to vorticity and it clearly is solenoidal because it is the curl of velocity.

Vector and scalar potential

The previous section showed that a vector field can be determined from the divergence and curl of the vector field and the values on surfaces of discontinuities and bounding surfaces. The integral equations are useful for developing analytical solutions for simple systems. However, in hydrodynamics the vorticity is generally a unknown quantity. Thus it is useful to express the potentials as differential equations that are solved simultaneously for the potentials and vorticity. The differential equations are derived by substituting the potentials into the expressions for the divergence and curl.

$$\nabla^2 \varphi = -4\pi \quad \rho$$
$$\nabla^2 \mathbf{A} = -4\pi \quad \mathbf{I}$$

In two-dimensional vector fields the vector potential **A** and the vector **I** has a nonzero component only in the third direction. In hydrodynamics the nonzero component of the vector potential is the stream function $4\pi\mathbf{I}$ corresponds to the vorticity, which has only one nonzero component.

Assignment 3.1: Particle velocity and acceleration

Suppose a particle fixed on the surface of a steady-rotating sphere with radius, *R*, has a constant speed, $|v|$=*v*.

a) Show that its acceleration is perpendicular to its velocity.
b) Show that the acceleration has a radial (from the center of the sphere) component $a_\rho = -v^2 / R$.
c) Let the magnitude of the angular velocity be ω. Express the centrifugal acceleration (direction perpendicular to the axis of rotation), a_r in terms of *R*, ω and the angle of particle from the axis of rotation.

Assignment 3.2: Differential area and volume

a) Express the differential of area in term of the differential of the surface coordinates for a spherical surface using the spherical polar coordinate system.
b) Obtain the differential volume elements in cylindrical and spherical polars by the Jacobian and check with a simple geometrical picture.

Assignment 3.3: Differential operators

a) Derive the expression for the Laplacian of a scalar in Cartesian coordinates from the definition of the gradient and divergence.
b) Prove that: $\nabla \bullet (\varphi \mathbf{a}) = \nabla \varphi \bullet \mathbf{a} + \varphi (\nabla \bullet \mathbf{a})$
c) Let **x** be the Cartesian coordinates of points in space and *r* = |**x**|. Calculate the divergence and curl of **x** and the gradient and Laplacian of *r* and *1/r*. Note any singularities.
d) Prove the identities involving the curl operator.
e) Suppose a rigid body has the velocity field $\mathbf{v} = \mathbf{v}_{(t)} + \omega \times (\mathbf{x} - \mathbf{x}_o)$. Show that the curl of this velocity field is $\nabla \times \mathbf{v} = 2\omega$.