## Chapter 2 - Cartesian Vectors and Tensors: Their Algebra



**Reading Assignment:** Chapter 2 of Aris, Appendix A of BSL

The algebra of vectors and tensors will be described here with Cartesian coordinates so the student can see the operations in terms of its components without the complexity of curvilinear coordinate systems.

### Definition of a vector

Suppose $x_i$ , i.e., ($x_1, x_2, x_3$), are the Cartesian coordinates of a point *P* in a frame of reference, *0123*. Let $0\overline{1}\overline{2}\overline{3}$ be another Cartesian frame of reference with the same origin but defined by a rigid rotation. The coordinates of the point *P* in the new frame of reference is $\overline{x}_j$ where the coordinates are related to those in the old frame as follows.

$$\overline{x}_j = l_{ij} \quad x_i = l_{1j}x_1 + l_{2j}x_2 + l_{3j}x_3$$
$$x_i = l_{ij} \quad \overline{x}_j = l_{i1}\overline{x}_1 + l_{i2}\overline{x}_2 + l_{i3}\overline{x}_3$$

where $l_{ij}$ are the cosine of the angle between the old and new coordinate systems. Summation over repeated indices is understood when a term or a product appears with a common index.

*Definition*. A Cartesian vector, **a**, in three dimensions is a quantity with three components $a_1, a_2, a_3$ in the frame of reference *0123,* which, under rotation of the coordinate frame to $0\overline{1}\overline{2}\overline{3}$ , become components $\overline{a}_1, \overline{a}_2, \overline{a}_3$ , where

$$\overline{a}_j = l_{ij}a_i$$

**Examples of vectors**

In Cartesian coordinates, the length of the position vector of a point from the origin is equal to the square root of the sum of the square of the coordinates. The *magnitude* of a vector, **a**, is defined as follows.

$$|\mathbf{a}| = \left(a_i a_i\right)^{1/2}$$

A vector with a magnitude of unity is called a *unit vector*. The vector, **a**/|**a**|, is a unit vector with the direction of **a**. Its components are equal to the cosine of the angle between **a** and the coordinate axis. Some special unit vectors are the unit vectors in the direction of the coordinate axis and the normal vector of a surface.

**Scalar multiplication**

If $\alpha$ is a scalar and **a** is a vector, the product $\alpha$**a** is a vector with components, $\alpha a_i$, magnitude $\alpha$|**a**|, and the same direction as **a**.

**Addition of vectors – Coplanar vectors**

If **a** and **b** are vectors with components $a_i$ and $b_i$, then the sum of **a** and **b** is a vector with components, $a_i+b_i$.

The order and association of the addition of vectors are immaterial.

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$$
$$\left(\mathbf{a}+\mathbf{b}\right)+\mathbf{c}=\mathbf{a}+\left(\mathbf{b}+\mathbf{c}\right)$$

The subtraction of one vector from another is the same as multiplying one by the scalar (-1) and adding the resulting vectors.

If **a** and **b** are two vectors from the same origin, they are *colinear* or parallel if one is a linear combination of the other, i.e., they both have the same direction. If **a** and **b** are two vectors from the same origin, then all linear combination of **a** and **b** are in the same plane as **a** and **b**, i.,e., they are *coplanar*. We will prove this statement when we get to the triple scalar product.

**Unit vectors**

The unit vectors in the direction of a set of mutually orthogonal coordinate axis are defined as follows.

$$\mathbf{e}_{(1)} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \mathbf{e}_{(2)} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad \mathbf{e}_{(3)} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

The suffixes to **e** are enclosed in parentheses to show that they do not denote components. A vector, **a**, can be expressed in terms of its components, ($a_1$, $a_2$, $a_3$) and the unit vectors.

$$\mathbf{a} = a_1\,\mathbf{e}_{(1)} + a_2\,\mathbf{e}_{(2)} + a_3\,\mathbf{e}_{(3)}$$

This equation can be multiplied and divided by the magnitude of **a** to express the vector in terms of its magnitude and direction.

$$\begin{aligned}\mathbf{a} &= |\mathbf{a}|\left( \frac{a_1}{|\mathbf{a}|}\mathbf{e}_{(1)} + \frac{a_2}{|\mathbf{a}|}\,\mathbf{e}_{(2)} + \frac{a_3}{|\mathbf{a}|}\mathbf{e}_{(3)} \right) \\ &= |\mathbf{a}|\left( \lambda_1 \mathbf{e}_{(1)} + \lambda_2\,\mathbf{e}_{(2)} + \lambda_3\,\mathbf{e}_{(3)} \right)\end{aligned}$$

where $\lambda_i$ are the directional cosines of **a**.

A special unit vector we will use often is the normal vector to a surface, **n**. The components of the normal vector are the directional cosines of the normal direction to the surface.

**Scalar product – Orthogonality**

The *scalar product* (or *dot product*) of two vectors, **a** and **b** is defined as

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where $\theta$ is the angle between the two vectors. If the two vectors are perpendicular to each other, i.e., they are *orthogonal*, then the scalar product is zero. The unit vectors along the Cartesian coordinate axis are orthogonal and their scalar product is equal to the Kronecker delta.

$$\begin{aligned}\mathbf{e}_{(i)} \bullet \mathbf{e}_{(j)} &= \delta_{ij} \\ &= \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}\end{aligned}$$

The scalar product is commutative and distributive. The cosine of the angle measured from **a** to **b** is the same as measured from **b** to **a**. Thus the scalar product can be expressed in terms of the components of the vectors.

$$\begin{aligned}\mathbf{a} \bullet \mathbf{b} &= \left(a_1\,\mathbf{e}_{(1)} + a_2\,\mathbf{e}_{(2)} + a_3\,\mathbf{e}_{(3)}\right) \bullet \left(b_1\,\mathbf{e}_{(1)} + b_2\,\mathbf{e}_{(2)} + b_3\,\mathbf{e}_{(3)}\right) \\ &= a_i\,b_j \delta_{ij} \\ &= a_i\,b_i\end{aligned}$$

The scalar product of a vector with itself is the square of the magnitude of the vector.

$$\begin{aligned}\mathbf{a} \bullet \mathbf{a} &= |\mathbf{a}||\mathbf{a}| \cos 0 \\ &= |\mathbf{a}|^2 \\ \mathbf{a} \bullet \mathbf{a} &= a_i\, a_i \\ &= |\mathbf{a}|^2\end{aligned}$$

The most common application of the scalar product is the projection or component of a vector in the direction of another vector. For example, suppose **n** is a unit vector (e.g., the normal to a surface) the component of **a** in the direction of **n** is as follows.

$$\mathbf{a} \bullet \mathbf{n} = |\mathbf{a}| \cos \theta$$

**Directional Cosines for Coordinate Transformation**

The properties of the directional cosines for the rotation of the Cartesian coordinate reference frame can now be easily illustrated. Suppose the unit vectors in the original system is $\mathbf{e_{(i)}}$ and in the rotated system is $\overline{\mathbf{e}}_{(j)}$. The components of the unit vector, $\overline{\mathbf{e}}_{(j)}$, in the original reference frame is $l_{ij}$. This can be expressed as the scalar product.

$$\begin{aligned}&\overline{\mathbf{e}}_{\mathbf{(j)}} = l_{1j}\, \mathbf{e_{(1)}} + l_{2j}\, \mathbf{e_{(2)}} + l_{3j}\, \mathbf{e_{(3)}}\ , \quad j = 1,2,3 \\ &\mathbf{e_{(i)}} \bullet \overline{\mathbf{e}}_{\mathbf{(j)}} = l_{ij}\ , \quad i, j = 1,2,3\end{aligned}$$

Since $\overline{\mathbf{e}}_{(j)}$ is a unit vector, it has a magnitude of unity.

$$\overline{\mathbf{e}}_{\mathbf{(j)}} \bullet \overline{\mathbf{e}}_{\mathbf{(j)}} = 1 = l_{i(j)}\, l_{i(j)} = l_{1(j)}\, l_{1(j)} + l_{2(j)}\, l_{2(j)} + l_{3(j)}\, l_{3(j)}\ , \quad j = 1,2,3$$

Also, the axis of a Cartesian system are orthorgonal.

$$\mathbf{e_{(i)}} \bullet \mathbf{e_{(j)}} = \begin{cases} 0\,, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

thus

$$\mathbf{e_{(i)}} \bullet \mathbf{e_{(j)}} = \delta_{ij}$$

$$\overline{\mathbf{e}}_{(\mathbf{i})} \bullet \overline{\mathbf{e}}_{(\mathbf{j})} = l_{ki}\, l_{kj} = l_{1i}\, l_{1j} + l_{2i}\, l_{2j} + l_{3i}\, l_{3j}\ , \quad i, j = 1,2,3$$
$$= \delta_{ij}$$

**Vector Product**

The *vector product* (or *cross product*) of two vectors, **a** and **b**, denoted as **a**×**b**, is a vector that is perpendicular to the plane of **a** and **b** such that **a**, **b**, and **a**×**b** form a right-handed system. (i.e., **a**, **b**, and **a**×**b** have the orientation of the thumb, first finger, and third finger of the right hand.) It has the following magnitude where θ is the angle between **a** and **b**.

$$\left|\mathbf{a}\times\mathbf{b}\right| = \left|\mathbf{a}\right|\left|\mathbf{b}\right|\sin\theta$$

The magnitude of the vector product is equal to the area of a parallelogram two of whose sides are the vectors **a** and **b**.

Since the vector product forms a right handed system, the product **b**×**a** has the same magnitude but opposite direction as **a**×**b**, i.e., the vector product is not commutative,

$$\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}$$

The vector product of a vector with itself or with a parallel vector is zero or the null vector, i.e., **a**×**a**=**0**. A quantity that is the negative of itself is zero. Also, the angle between parallel vectors is zero and thus the sine is zero.

Consider the vector product of the unit vectors. They are all of unit length and mutually orthogonal so their vector products will be unit vectors. Remembering the right-handed rule, we therefore have

$$\begin{aligned}
\mathbf{e}_{(2)}\times\mathbf{e}_{(3)} &= -\mathbf{e}_{(3)}\times\mathbf{e}_{(2)} = \mathbf{e}_{(1)} \\
\mathbf{e}_{(3)}\times\mathbf{e}_{(1)} &= -\mathbf{e}_{(1)}\times\mathbf{e}_{(3)} = \mathbf{e}_{(2)} \\
\mathbf{e}_{(1)}\times\mathbf{e}_{(2)} &= -\mathbf{e}_{(2)}\times\mathbf{e}_{(1)} = \mathbf{e}_{(3)}
\end{aligned}$$

The components of the vector product can be expressed in terms of the components of **a** and **b** and applying the above relations between the unit vectors.

$$\begin{aligned}
\mathbf{a}\times\mathbf{b} &= \left(a_1\,\mathbf{e}_{(1)} + a_2\,\mathbf{e}_{(2)} + a_3\,\mathbf{e}_{(3)}\right)\times\left(b_1\,\mathbf{e}_{(1)} + b_2\,\mathbf{e}_{(2)} + b_3\,\mathbf{e}_{(3)}\right) \\
&= \left(a_2 b_3 - a_3 b_2\right)\mathbf{e}_{(1)} + \left(a_3 b_1 - a_1 b_3\right)\mathbf{e}_{(2)} + \left(a_1 b_2 - a_2 b_1\right)\mathbf{e}_{(3)}
\end{aligned}$$

The permutations of the indices and signs in the expression for the vector product may be difficult to remember. Notice that the expression is the same as that for the expansion of a determinate of the matrix,

$$\begin{vmatrix} \mathbf{e}_{(1)} & \mathbf{e}_{(2)} & \mathbf{e}_{(3)} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Expansion of determinants are aided by the permutation symbol, $\varepsilon_{ijk}$.

$$\varepsilon_{ijk} = \begin{cases} 0, \text{ if any two of } i, j, k \text{ are the same} \\ +1, \text{ if } ijk \text{ is an even permutation of } 1, 2, 3 \\ -1, \text{ if } ijk \text{ is an odd permutation of } 1, 2, 3 \end{cases}$$

The expression for the vector product is now as follows.

$$\mathbf{a} \times \mathbf{b} = \varepsilon_{ijk}\, a_i\, b_j\, \mathbf{e}_{(k)}$$

**Velocity due to rigid body rotations**

We will show that the velocity field of a rigid body can be described by two vectors, a translation velocity, $\mathbf{v}_{(t)}$, and an angular velocity, $\boldsymbol{\omega}$. A rigid body has the constraint that the distance between two points in the body does not change with time. The translation velocity is the velocity of a fixed point, *O*, in the body, e.g., the center of mass. Now consider a new reference frame (coordinate system) with the origin at point *O* that is translating with respect to the original reference frame with the velocity $\mathbf{v}_{(t)}$. The rotation of the body about *O* is defined by the angular velocity, $\boldsymbol{\omega}$, i.e., with a magnitude $\omega$ and a direction of the axis of rotation, **n**, such that the positive direction is the direction that a right handed screw advances when subject to the rotation, . $\boldsymbol{\omega}=\omega\, \mathbf{n}$. Consider a point *P* not on the axis of rotation, having coordinates **x** in the new reference frame. The velocity of *P* in the new reference frame has a magnitude equal to the product of $\omega$ and the radius of the point *P* from the axis of rotation. This radius is equal to the magnitude of **x** and the sine of the angle between **x** and $\boldsymbol{\omega}$, i.e., $|\mathbf{x}| \sin\theta$. The velocity of point *P* in the new reference frame can be expressed as

$$\mathbf{v} = \omega \times \mathbf{x}$$
$$|\mathbf{v}| = \omega |\mathbf{x}| \sin\theta$$

The velocity field of any point of the rigid body in the original reference frame is now

$$\mathbf{v} = \mathbf{v}_{(t)} + \omega \times \left(\mathbf{x} - \mathbf{x}_o\right)$$

where $\mathbf{x}_o$ is the coordinates of point *O* in the original reference frame. Since this equation is valid for any pair of points in the rigid body, the relative velocity $\Delta\mathbf{v}$ between a pair of points separated by $\Delta\mathbf{x}$ can be expressed as follows.

$$\Delta \mathbf{v} = \omega \quad \times \Delta \mathbf{x}$$

Conversely, if the relative velocity between any pair of points is described by the above equation with the same value of angular velocity, then the motion is due to a rigid body rotation.

**Triple scalar product**

The triple scalar product is the scalar product of the first vector with the vector product of the other two vectors. It is denoted as (**abc**) or [**abc**].

$$(\mathbf{abc}) \equiv \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$$

Recall that **b**×**c** has a magnitude equal to the area of a parallelogram with sides **b** and **c** and a direction normal to the plane of **b** and **c**. The scalar product of this normal vector and the vector **a** is equal to the altitude of the parallelepiped with a common origin and sides **a**, **b**, and **c**. The triple scalar product has a magnitude equal to the volume of a parallelepiped with a common origin and sides **a**, **b**, and **c**. The sign of the triple scalar product can be either positive or negative. If **a**, **b**, and **c** are coplanar, then the altitude of the parallelepiped is zero and thus the triple scalar product is zero.

The triple scalar product can be expressed in terms of the components by using the earlier definitions of the vector product and scalar product.

$$\begin{aligned}
\mathbf{b} \times \mathbf{c} &= \varepsilon_{ijk}\, b_i\, c_j\, \mathbf{e}_{(k)} \\
\mathbf{a} &= a_m \quad \mathbf{e}_{(m)} \\
\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) &= \varepsilon_{ijk} a_m\, b_i\, c_j\, \mathbf{e}_{(m)} \bullet \mathbf{e}_{(k)} = \varepsilon_{ijk} a_m\, b_i\, c_j\, \delta_{mk} = \varepsilon_{ijk} a_k\, b_i\, c_j \\
&= \varepsilon_{ijk} a_i\, b_j\, c_k
\end{aligned}$$

From the definition of the permutation symbol, the triple scalar product is unchanged by even permutations of **a**, **b**, and **c** but have the opposite algebraic sign for odd permutations. Also, if any two of **a**, **b**, and **c** are identical, then permutation of the two identical vectors results in a triple scalar products that are identical and also opposite in sign. This implies that the triple scalar product is zero if two of the vectors are identical.

**Triple vector product**

The triple vector product of vectors **a**, **b**, and **c** results from the repeated application of the vector product, i.e., **a**×(**b**×**c**). Since **b**×**c** is normal to the plane of **a** and **b** and **a**×(**b**×**c**) is normal to **b**×**c, a**×(**b**×**c**) must be in the plane of **a** and **b**. It is left as an exercise to show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$$

**Second order tensors**

A second order tensor can be written as a 3×3 matrix.

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

A tensor is a physical entity that is the same quantity in different coordinate systems. Thus a second order tensor is defined as an entity whose components transform on rotation of the Cartesian frame of reference as follows.

$$\overline{A}_{pq} = l_{ip}\, l_{jq}\, A_{ij}$$

If $A_{ij}=A_{ji}$ the tensor is said to be *symmetric* and a symmetric tensor has only six distinct components. If $A_{ij}=-A_{ij}$ the tensor is said to be *antisymmetric* and such a tensor is characterized by only three nonzero components for the diagonal terms, $A_{ii}$, are zero. The tensor whose $ij^{th}$ element is $A_{ji}$ is called the transpose **A'** of **A**. The determinant of a tensor is the determinant of the matrix of its components.

$$\det \mathbf{A} = \varepsilon_{ijk}\, A_{1i}\, A_{2j}\, A_{3k}$$

**Examples of second order tensors**

A second order tensor we have already encountered is the *Kronecker delta* $\delta_{ij}$. Of its nine components, the six off-diagonal components vanish and the three diagonal components are equal to unity. It transforms as a tensor upon transforming its components to a rotated frame of reference.

$$\begin{aligned} \overline{\delta}_{pq} &= l_{ip}\, l_{jq}\, \delta_{ij} \\ &= l_{ip}\, l_{iq} \\ &= \delta_{pq} \end{aligned}$$

because of the orthogonality relation between the directional cosines $l_{ij}$. In fact, the components of $\delta_{ij}$ in all coordinates remain the same. $\delta_{ij}$ is called the isotropic tensor for that reason. The transport coefficients (e.g., thermal conductivity) of an isotropic medium can be expressed as a scalar quantity multiplying $\delta_{ij}$.

If a and b are two vectors, the set of nine products, $a_i b_j = A_{ij}$, is a second order tensor, for

$$\begin{aligned} \overline{A}_{pq} &= \overline{a}_p\, \overline{b}_q = l_{ip}\, a_i\, l_{jq}\, b_j = l_{ip}\, l_{jq} \left( a_i\, b_j \right) \\ &= l_{ip}\, l_{jq}\, A_{ij} \end{aligned}.$$

An important example of this is the momentum flux tensor. If $\rho$ is the density and **v** is the velocity, $\rho v_i$ is the $i^{th}$ component in the direction $Oi$. The rate at which this momentum crosses a unit area normal to $Oj$ is the tensor, $\rho v_i v_j$.

**Scalar multiplication and addition**

If $\alpha$ is a scalar and **A** a second order tensor, the scalar product of $\alpha$ and **A** is a tensor $\alpha$**A** each of whose components is $\alpha$ times the corresponding component of **A**.

The sum of two second order tensors is a second order tensor each of whose components is the sum of the corresponding components of the two tensors. Thus the $ij^{th}$ component of **A**+**B** is $A_{ij} + B_{ij}$. Notice that the tensors must be of the same order to be added; a vector can not be added to a second order tensor. A linear combination of tensors results from using both scalar multiplication and addition. $\alpha$**A** + $\beta$**B** is the tensor whose $ij^{th}$ component is $\alpha A_{ij} + \beta B_{ij}$. Subtraction may therefore be defined by putting $\alpha = 1$, $\beta = -1$.

Any second order tensor can be represented as the sum of a symmetric part and an antisymmetric part. For

$$A_{ij} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) + \frac{1}{2}\left(A_{ij} - A_{ji}\right)$$

and changing $i$ and $j$ in the first factor leaves it unchanged but changes the sign of the second. Thus,

$$\mathbf{A} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A'}\right) + \frac{1}{2}\left(\mathbf{A} - \mathbf{A'}\right)$$

represents **A** as the sum of a symmetric tensor and antisymmetric tensor.

**Contraction and multiplication**

As in vector operations, summation over repeated indices is understood with tensor operations. The operation of identifying two indices of a tensor and so summing on them is known as *contraction*. $A_{ii}$ is the only contraction of $A_{ij}$,

$$A_{ii} = A_{11} + A_{22} + A_{33}$$

and this is no longer a tensor of the second order but a scalar, or a tensor of order zero. The scalar $A_{ii}$ is known as the trace of the second order tensor **A**. The notation tr **A** is sometimes used. The contraction operation in computing the trace of a tensor **A** is analogous to the operation in the calculation the magnitude of a vector **a**, i.e., .$|\mathbf{a}|^2 = \mathbf{a}\bullet\mathbf{a} = a_i a_i = a_1 a_1 + a_2 a_2 + a_3 a_3$.

If **A** and **B** are two second order tensors, we can form 81 numbers from the products of the 9 components of each. The full set of these products is a fourth order tensor. Contracted products result in second order or zero order tensors. We will not have an occasion to use products of tensors in our course.

The product $A_{ij}\ a_j$ of a tensor **A** and a vector **a** is a vector whose $i^{th}$ component is $A_{ij}\ a_j$. Another possible product of these two is $A_{ij}\ a_l$ . These may be written **A•a** and **a•A** , respectively. For example, the diffusive flux of a quantity is computed as the contracted product of the transport coefficient tensor and the potential gradient vector, e.g., $\mathbf{q} = -\mathbf{k}\bullet\nabla T$.

**The vector of an antisymmetric tensor**

We showed earlier that a second order tensor can be represented as the sum of a symmetric part and an antisymmetric part. Also, an antisymmetric tensor is characterized by three numbers. We will later show that the antisymmetric part of the velocity gradient tensor represents the local rotation of the fluid or body. Here, we will develop the relation between the angular velocity vector, $\boldsymbol{\omega}$, introduced earlier and the corresponding antisymmetric tensor.

Recall that the relative velocity between a pair of points in a rigid body was described as follows.

$$\Delta\mathbf{v} = \omega \times \Delta\mathbf{x}$$

We wish to define a tensor $\boldsymbol{\Omega}$ that also can determine the relative velocity.

$$\begin{aligned}\Delta\mathbf{v} &= \omega \times \Delta\mathbf{x}\\ &= \Delta\mathbf{x} \bullet \Omega\end{aligned}$$

The following relation between the components satisfies this relation.

$$\begin{aligned}\Omega_{ij} &= \varepsilon_{ijk}\ \omega_k\\ \omega_k &= \frac{1}{2}\varepsilon_{ijk}\ \Omega_{ij}\end{aligned}$$

Written in matrix notation these are as follows.

$$\omega = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix},\ \Omega = \begin{bmatrix}0 & \omega_3 & -\omega_2\\ -\omega_3 & 0 & \omega_1\\ \omega_2 & -\omega_1 & 0\end{bmatrix}$$

The notation **vec** $\boldsymbol{\Omega}$ is sometimes used for $\boldsymbol{\omega}$. In summary, an antisymmetric tensor is completely characterized by the vector, **vec** $\boldsymbol{\Omega}$.

**Canonical form of a symmetric tensor**

We showed earlier that any second order tensor can be represented as a sum of a symmetric part and an antisymmetric part. The symmetric part is determined by 6 numbers. We now seek the properties of the symmetric part. A theorem in linear algebra states that a symmetric matrix with real elements can be transformed by its eigenvectors to a diagonal matrix with real elements corresponding the eigenvalues. (see Appendix A of Aris.) If the eigenvalues are distinct, then the eigenvector directions are orthogonal. The eigenvectors determine a coordinate system such that the contracted product of the tensor with unit vectors along the coordinate axis is a parallel vector with a magnitude equal to the corresponding eigenvalue. The surface described by the contracted product of all unit vectors in this transformed coordinate system is an ellipsoid with axes corresponding to the coordinate directions.

The eigenvalues and the scalar invariants of a second order tensor can be determined from the characteristic equation.

$$\det\left(A_{ij} - \lambda\delta_{ij}\right) = \Psi - \lambda\,\Phi + \lambda^2\,\Theta - \lambda^3$$

where

$$\Theta = A_{11} + A_{22} + A_{33} = tr\,\mathbf{A}$$

$$\Phi = A_{22}\,A_{33} - A_{23}\,A_{32} + A_{33}\,A_{11} - A_{31}\,A_{13} + A_{11}\,A_{22} - A_{12}\,A_{21}$$

$$\Psi = \det\mathbf{A}$$

Assignment 2.1

a) Relative velocity of points in a rigid body. If **x** and **y** are two points inside a rigid body that is translating and rotating, determine the relation between the relative velocity of these two points as a function of their relative positions. If **x** and **y** are points on a line parallel to the axis of rotation, what is their relative velocity? If **x** and **y** are points on opposite sides of the axis of rotation but with equal distance, r, what is their relative velocity? Draw diagrams.
b) Prove that: $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}$
c) Show $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$ vanishes identically if two of the three vectors are proportional of one another.
d) Show that if **a** is coplanar with **b** and **c**, then $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$ is zero.
e) Prove that: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\bullet\mathbf{c})\;\mathbf{b} - (\mathbf{a}\bullet\mathbf{b})\;\mathbf{c}$
f) Prove that the contracted product of a tensor **A** and a vector **a**, $\mathbf{A}\bullet\mathbf{a}$, transforms under a rotation of coordinates as a vector.
g) Show that you get the same result for relative velocity whether you use $\boldsymbol{\omega}$ or $\boldsymbol{\Omega}$ for the rotation of a rigid body.