The Gyroscope

"A large acquaintance with particulars often makes us wiser than the possession of abstract formulas, however deep."

W. James

OBJECTIVES

To qualitatively observe some of the motions of a gyroscope. To quantitatively verify a simple motion.

THEORY

A gyroscope is defined as a rigid rotating object, symmetric about one axis. Generations of children, back at least to Greek antiquity, have found fascination in the behavior of tops, to give the gyroscope its common name. A number of eminent physicists have also found the complex behavior of spinning objects a matter of interest and a fit subject for detailed analysis. More recently, very carefully engineered gyroscopes were used for navigation because the axis of spin points in a nearly fixed direction when external torques are small. This makes the gyroscope a good replacement for a magnetic compass, particularly in regions where magnetic compasses are unreliable.

As with any mechanical system, the motion of a gyroscope can be understood completely by a systematic application of $\vec{F} = m\vec{a}$ to all the particles of which the rigid body is made. It is much more efficient, however, to exploit the fact that most of the forces act between the particles of the body, and simply have the effect of making it rigid. The overall motion is then described by

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{1}$$

with $\vec{\tau} = \vec{r} \times \vec{F}$ the torque due to external forces $\vec{F}$. Although an apparently simple equation, analysis of the resulting motions can become very complicated.

We will simplify matters by considering the somewhat idealized example sketched in Fig. 1. Our toy is spinning about its axis with an angular speed ω, supported at one end on a frictionless bearing. Choosing the origin at the pivot, gravity will produce a torque about the origin because the center of mass is not necessarily above the pivot point, but there are no other external forces that can produce a torque because the bearing is assumed frictionless. This implies that both L_z and ω must be constant. Further, the total mechanical energy, including
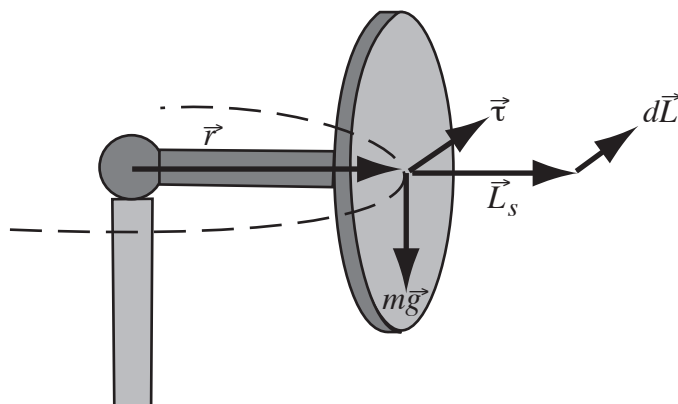
Fig. 1 A simple gyroscope, showing precession.

gravitational potential, must also be constant. The motion will still be interesting, but these conditions let us understand some qualitative features.

First, consider the case where the top is spinning rapidly with its axis more or less horizontal. The external force is vertically downwards, so the torque is horizontal, perpendicular to the axis of rotation. Since the spin angular momentum $\vec{L}_s$ is parallel to the axis of rotation, Eq. 1 tells us that $d\vec{L}$ is perpendicular to $\vec{L}_s$, and to a first approximation $\vec{L}_s + d\vec{L}$ is the same length as $\vec{L}_s$ but pointing in a different direction. In fact, as the motion proceeds the tip of $\vec{L}_s$ traces out a circle at a constant angular speed Ω, as indicated in Fig. 1. This motion is called precession, and we will demonstrate below that it follows from the equations of motion.

To start the gyroscope, we will hold the axis fixed and set the rate of spin to the desired value. If we then move the axis at the precession speed and release it, the motion will be a smooth precession. If, instead, the axis is released from rest the tip will trace out small 'scallop' or looping motions, superimposed on the overall precession. This is called nutation, and arises from conservation of mechanical energy. The precessional motion represents additional kinetic energy, relative to the state with the axis fixed. Since ω is constant (frictionless bearing), the additional kinetic energy must come from a loss of gravitational potential. In other words, the center of mass must fall a little bit, tipping the axis of rotation, in order for the top to precess. If the spin is rapid, the drop is small, and the precession is affected only slightly. Overall, the tip of the axis bounces up and down a little, and the precessional speed varies a little. If the spin is not fast enough the character of the motion changes drastically, but that is a complicated story.

For a quantitative comparison with Eq. 1, we can derive the angular frequency Ω of the steady precession, assuming there is no nutation. In that situation we can write the total angular momentum as the sum of the spin and precession angular momenta

$$\vec{L} = \vec{L}_s + \vec{L}_z = I\omega\left(\cos\theta\sin\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\phi\hat{k}\right) + I_z\Omega\hat{k} \quad (2)$$
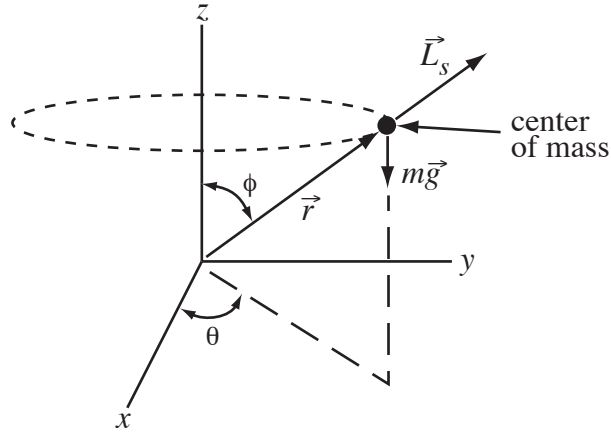
Fig. 2 Coordinates for analyzing gyroscopic motion.

where I_z is the moment of inertia about the vertical, and the angles are shown in Fig. 2. For steady precession, ϕ is constant and $d\theta / dt = \Omega$, so

$$\frac{d\vec{L}}{dt} = I\omega\left(-\Omega\sin\theta \, \sin\phi \hat{i} + \Omega\cos\theta\sin\phi \hat{j}\right). \tag{3}$$

The vector from the pivot to the center of mass is

$$\vec{r} = r\cos\theta\sin\phi \hat{i} + r\sin\theta\sin\phi \hat{j} + r\cos\phi \hat{k} \tag{4}$$

so the gravitational torque is

$$\vec{\tau} = \vec{r} \times m\vec{g} = -rmg\sin\theta\sin\phi \hat{i} + rmg\cos\theta\sin\phi \hat{j}. \tag{5}$$

These will satisfy Eq. 1 if

$$\Omega = \frac{rmg}{I\omega}, \tag{6}$$

demonstrating that uniform precession is a possible solution to the equations of motion. It is also amusing to note that Ω is independent of the tilt angle, perhaps an unexpected result.

EXPERIMENTAL PROCEDURE

1. Physical arrangement

    The gyroscope we will use is a solid metal sphere supported on a cushion of air, as sketched in Fig. 3. The air cushion effectively supports the sphere under its geometric center, so if the sphere were perfect there would be no torques acting at all. The sphere actually has a rod fastened to it, which serves to displace the center of mass away from the geometric center of the sphere and allow gravity to exert a small torque on the system. A sliding weight can be fastened to the rod to adjust the torque as desired. The rod also serves as a convenient marker for the axis of rotation, and a handle for manipulating the gyroscope. A secondary jet of air is available to spin the sphere very rapidly.

    There are a number of precautions which you must be aware of to avoid problems for yourself or the apparatus. The rod is strong enough to support the heavy sphere only in the vertical position, so avoid picking up the sphere by the rod if at all possible. The sphere itself is rather soft metal and it must be exactly round for the experiment to work. It will, therefore, be ruined if it falls onto the hard floor. Finally, you will be working with a rapidly spinning object. If the spinning ball contacts the supporting cup it will probably climb out and proceed across the room, damaging itself and perhaps also you. You can avoid this problem by leaving the air supply on whenever the sphere is spinning, and by not pushing the sphere into the support. The rod has a bearing on the end which you can hold to control the sphere without slowing it down. When you do want to slow the rotation, gently squeeze the rod with two fingers, while holding onto the bearing to steady it. Do not push directly on the sphere. With a little care and common sense you should not have any problems.

2. Qualitative observations

    Begin your work by turning on the air supply so that the sphere floats. You may have to lift the sphere slightly to start the flow. If the sliding weight was left on the rod, set it aside for later use. Holding the rod vertically with your fingers on the bearing, twirl the rod between the fingers of your other hand to start the sphere spinning. You can now tilt the rod to any desired
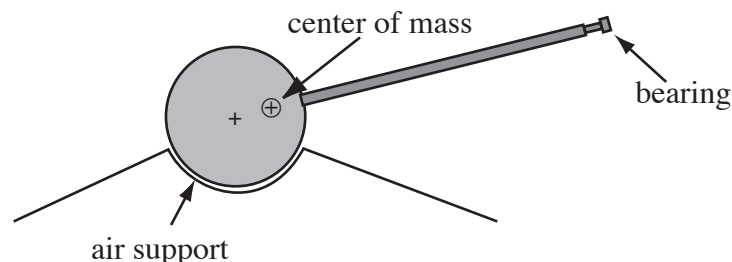

Fig.3. Cut-away view of the air-supported gyroscope

angle and release it to observe the precession and nutation.

Try this for a few different angles. Try also to see the effects of releasing the rod from rest, when moving at about the precession speed, and when moving in the opposite direction. It may help to think of the tip as marking a point on the surface of a sphere. The precession then corresponds to tracing a circle of constant latitude and the nutation is a small deviation from the circular path. The resulting motion is complicated, so this will be a real challenge to your observational skills. Describe the motion of the tip of the rod for various conditions as best you can, perhaps with the aid of a sketch.

As noted in the theory section, the motion of a slow top is quite distinctive. You can demonstrate this by spinning the sphere with the rod vertical and then releasing it. Start out with a rapid rotation, for which the motion is a slow precession, and then slow down the rotation by gently grasping the rod between your fingers while holding onto the bearing. Decrease the rate of spin in several stages, releasing the rod to observe the motion after each decrease. Describe the motion. Is it still reasonable to approximate the motion as a uniform precession plus a small nutation?

3. Quantitative analysis

The quantitative part of this experiment consists of checking two of the predictions of Eq. 6. We expect Ω to depend directly on rmg, and inversely on ω. Since we can vary both of these, we can check the prediction. For an accurate test we need to spin the sphere quite rapidly, so we will use an air jet. Hold the rod horizontal, grasping just the bearing, and direct the air at the equator of the sphere. After three or four minutes the sphere will be spinning quite rapidly. If you now set the jet aside and release the rod, the gyroscope will precess slowly. You can measure the time for a complete precession with a stopwatch, using some convenient reference for the start and stop points. Of course, the rate of spinning will decrease with time, but the friction is small enough that this process is slow.

To measure the spin velocity we will use a strobe light. Position the strobe so that it will brightly illuminate the sphere, but not so close that it will interfere with the motion. When the interval between flashes is exactly the same as the time for one rotation of the sphere, the rotational motion will appear to be "stopped", because the same parts of the sphere will be in the same place at each flash. The sphere has been marked with two black lines to make it easier to tell when it is "stopped". By measuring the time interval between flashes when this condition is met, we determine the period of the spinning motion.

Getting the flash rate correct is a little tricky at first. If the strobe is flashing too often, the sphere will not make a complete revolution between flashes, and you will see multiple images of the black markings. Alternatively, if the strobe is flashing too slowly you may see only one

image, but the mark may actually be illuminated only every second or third revolution of the sphere. A useful tactic, therefore, is to start with a low flash rate and increase until you find the fastest rate at which you see only one image. The strobe should then be flashing exactly once per revolution of the sphere. As a check, you can exactly double the rate, if that is within the range of the strobe. You should then see the images of the two black lines superimposed in an "X".

The flash rate is displayed on the strobe in flashes per minute. You will need to convert to the time interval between flashes, seconds per flash, to obtain the spin period T_s.

To test Eq. 6 you would measure the precession velocity for several values of spin velocity. However, the quantities you determine directly are the precession and spin periods, T_p and T_s rather than the angular velocities Ω and ω. Since $\Omega = 2\pi/T_p$, and correspondingly for ω, we can rewrite Eq. 6 as

$$T_p T_s = \frac{4\pi^2 I}{rmg} \tag{7}$$

This says that the product of T_p and T_s should be a constant, about 1.0 s^2 for our apparatus, independent of ω. Do your measurements support this claim?

Using the sliding weight we can also check the dependence of the precession period on rmg. An additional weight m', placed on the rod a distance r' from the center of the sphere will simply add an additional term in the denominator of Eq. 7, giving

$$T_p T_s = \frac{4\pi^2 I}{rmg + r'm'g} \tag{8}$$

which is more informative if both sides are inverted

$$\frac{1}{T_p T_s} = \frac{g}{4\pi^2 I}\left(rm + r'm'\right) \tag{9}$$

A plot of $(T_p T_s)^{-1}$ vs r' should evidently be a straight line, with an intercept at the value found with no additional weight. To check this idea, slip the weight onto the rod at some convenient place and measure r'. (Measure from the surface of the sphere to the middle of the weight and add 2.5 cm to account for the radius of the sphere.) Determine T_s and T_p for several different values of r', stopping the sphere to adjust the weight each time. Is your plot straight? Does it have the right intercept?