Physics 102 Spring 2003: Test 1—Free Response and Instructions

- Print your LAST and FIRST name on the front of your blue book, on this question sheet, the multiple-choice question sheet and the multiple-choice answer sheet.
- TIME ALLOWED 90 MINUTES
- The test consists of two free-response questions and ten multiple-choice questions.
- The test is graded on a scale of 100 points; the first free-response question accounts for 40 points, the second free-response question accounts for 30 points, and the multiple-choice questions account for 30 points.
- Answer the two free-response questions in your blue book. Answer the multiple-choice questions by marking a dark X in the appropriate column and row in the table on the multiple-choice answer sheet.
- Consult no books or notes of any kind. You may use a hand-held calculator in non-graphing, non-programmed mode.
- Do NOT take test materials outside of the class at any time. Return this question sheet along with your blue book and multiple-choice question sheet.
- Write and sign the Pledge on the front of your blue book.

Show your work for the free-response problems, including neat and clearly labelled figures, in your blue book. Answers without explanation (even correct answers) will not be given credit.

I. (40 pts) A model of the electron treats it as a sphere of radius R_e with the charge $-e$ uniformly distributed throughout its volume. The coordinate r measures the distance from the center of the sphere. Express your answers in terms of $e, R_e, \epsilon_0,$ and r.

(a) Determine the electric field $\vec{E}(r)$ for all values of r.
(b) Taking the zero of the electrostatic potential to be at infinity, determine $V(r)$ for all values of r.
(c) Sketch $V(r)$ for all r.
(d) Determine the energy density in the electric field $u_E(r)$ for all values of r.
(e) Determine the total energy stored in the electric field by integrating the energy density $u_E(r)$ over all space.
(f) A classical model of the electron equates the rest mass energy of the electron $m_e c^2$ to the total energy stored in the electric field. From your result in (e) determine this so-called classical radius of the electron R_e in terms of the charge of the electron e, the rest mass energy of the electron $m_e c^2$, and other physical constants.

We know from direct measurements of the electron size in scattering experiments that in fact the electron is much smaller than this classical model would indicate. So, in fact, the rest mass energy of the electron must have some other origin than the energy stored in the electric field.

II. (30 pts) Two positive point charges $+Q$ are located on the y-axis at $y = \pm a$ as shown below.
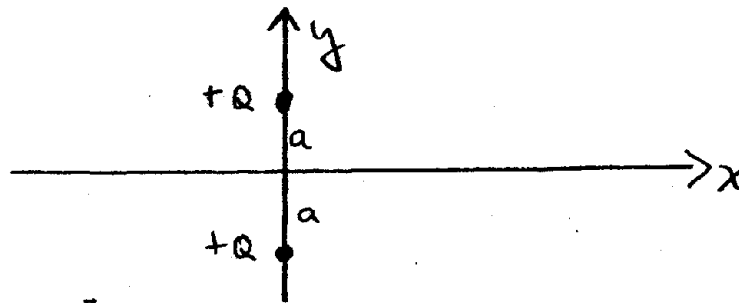
(a) Determine the electric field $\vec{E}$ on the x-axis, for all values of x.
(b) Determine the electrostatic potential $V(x)$ for all points on the x-axis.
(c) Sketch $V(x)$ for all values of x on the x-axis.
(d) From your plot in (c), determine at which points on the x-axis the electric field is zero.

Now suppose a third positive charge of magnitude $+2Q$ is brought in from infinity and placed at the origin.
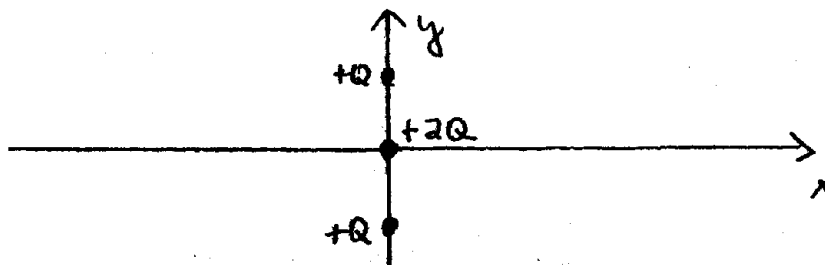
(e) How much work must be done by an external agent to put the $+2Q$ charge at the origin?
(f) What is the total electrostatic energy of this three-charge configuration?
(g) If all three charges are released and allowed to move freely, what is the sum of their kinetic energies when all the particles are very far from each other?

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1. A cylindrical capacitor is constructed so that the space between the conducting cylinders is filled with an insulating fluid. The capacitor is attached to a battery so that the inner and outer conducting cylinders are held at a fixed potential difference. Then, while still connected to the battery, the fluid begins to leak out. (Oops!) Which of the following statements is (are) true after all the fluid has leaked out?

   I.   The electric field in the region between the inner and outer cylinders has decreased.
   II.   The potential difference between the inner and outer cylinders has decreased.
   III. The charge stored on the inner and outer cylinders has decreased.

   a.   Only one of these statements is true.
   b.   Only I and II are true.
   c.   Only II and III are true.
   d.   Only I and III are true.
   e.   I, II and III are all true.

2. Consider the three dipoles depicted below. If a uniform external electric field which points in the $+\hat{\jmath}$ direction (i.e. toward the top of the page) is turned on, which dipole will experience the torque of largest magnitude and which dipole will have the greatest potential energy?
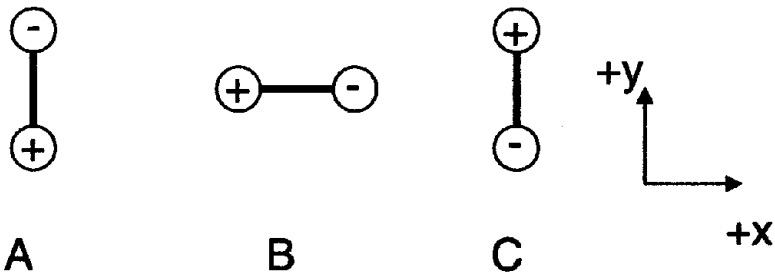

   a.   Greatest magnitude of torque on A; greatest potential energy for B.
   b.   Greatest magnitude of torque on B; greatest potential energy for A.
   c.   Greatest magnitude of torque on B; greatest potential energy for C.
   d.   Greatest magnitude of torque on A; greatest potential energy for C.
   e.   Greatest magnitude of torque on B; greatest potential energy for C.
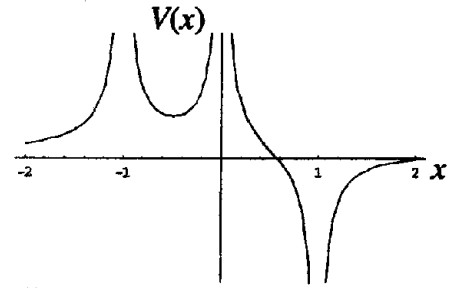
3. In which of the charge configurations depicted below is the magnitude of the force on an electron placed at the point in the center of the square the greatest?
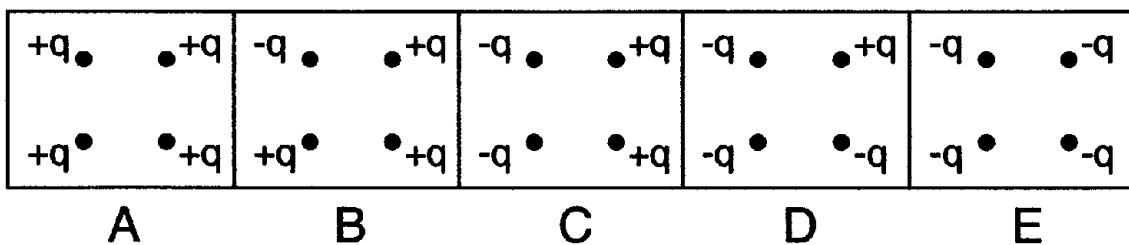
4. Three charges of equal magnitude are placed at the corners of an equilateral triangle. One charge has a sign opposite to the other two. Which of the graphs below correctly depicts the equally-spaced equipotential surfaces in the plane of the triangle? (All graphs to same scale.)
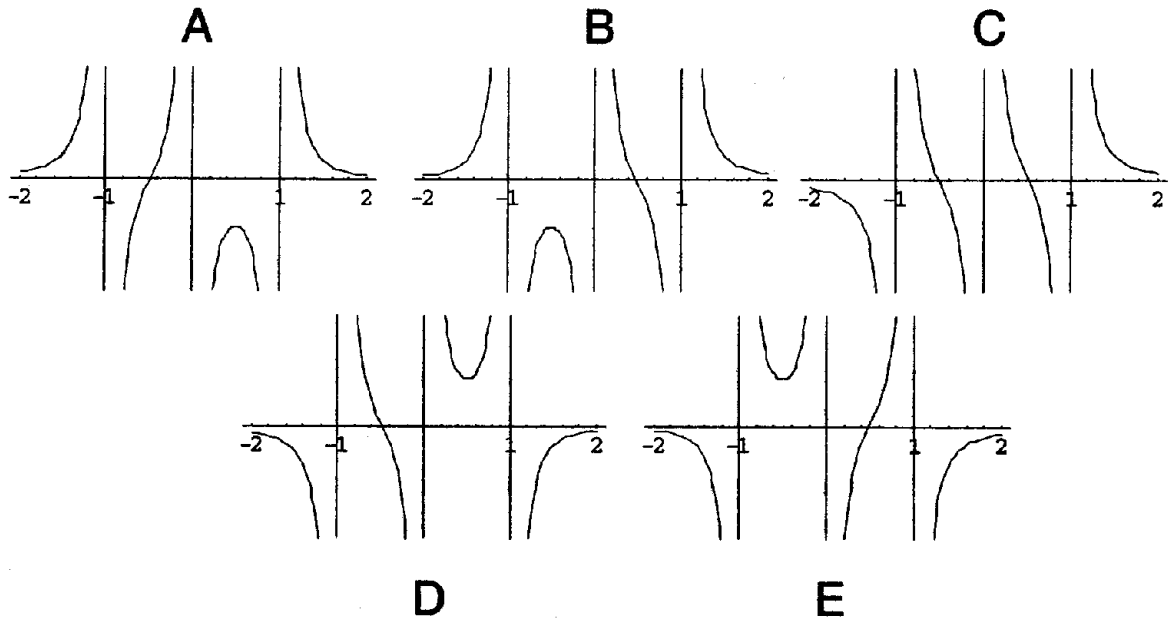

A   B   C   D   E
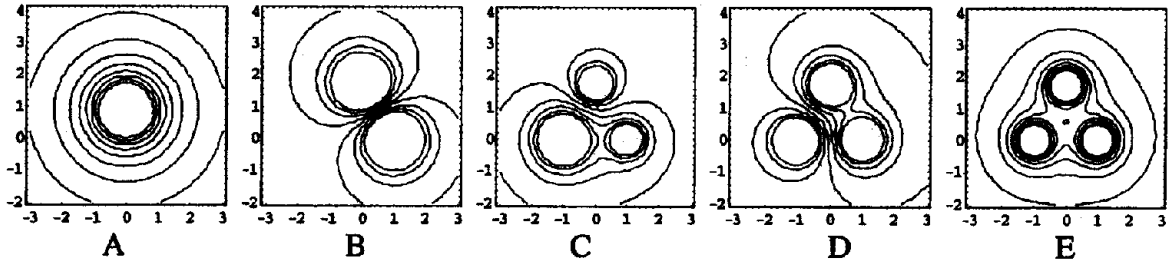
For the questions 5 and 6, refer to the graph to the right, depicting the potential on the x-axis as a function of x.


5. Which of the charge configurations depicted below would give rise to that potential?

A   B   C   D   E
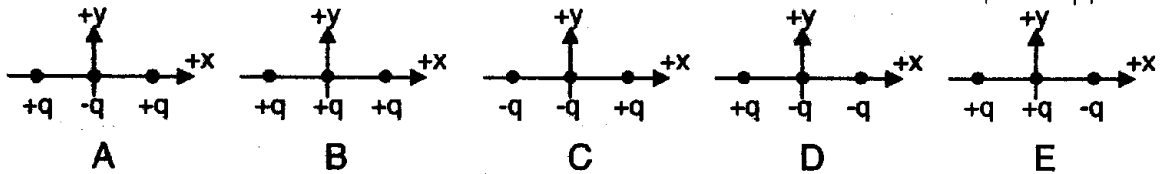
6. What is the x-component of the electric field for points on the x-axis given the potential depicted above?

A   B   C

D   E

7. Consider the capacitor circuit depicted to the right. A potential difference is applied across points **a** and **b**. If $C_3=2C_2=4C_1$, which capacitor(s) carries(y) the most charge?
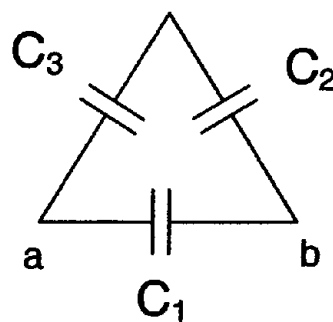

a. C_1
b. C_2
c. C_3
d. C_2 and C_3 carry the same amount that is greater than the charge on C_1
e. All carry the same charge.

8. An object made of an ideally conducting material with the shape depicted to the right is isolated from other charges and fields and given a positive net charge $+Q$. Which of the following statements are true?

a. The potential of the conductor is greatest on the surface of the pointy end.
b. The magnitude of the electric field is greatest just outside the surface of the pointy end.
c. The work required to add another charge to the conductor is greatest for a charge added to the pointy end.
d. The capacitance of the conductor is greatest at the surface of the pointy end.
e. The volume charge density is greatest for the material in the pointy end.

9. Consider a parallel pair of plates, one charged $+Q$, the other $-Q$. The area of the plates is A and their separation is d. Compare this arrangement to a similar arrangement of plates where the scale of the geometry has been increased by a factor of two but the amount of charge on the plates has remained fixed. In other words, lengths are increases by a factor of two, areas by a factor of four, etc. Which of the following is a true statement about the energy densities of the electric fields in the different arrangements?

a. $u_{little} = u_{big}$
b. $u_{little} = 2u_{big}$
c. $u_{little} = 4u_{big}$
d. $u_{little} = 8u_{big}$
e. $u_{little} = 16u_{big}$

10. Two charges Q and q, separated by a distance d, produce a potential $V = 0$ at a point P somewhere on the line that connects them. (Note: the potential at infinity is zero in this question.) Which of the following statements is true?

a. No force would act on a test charge place at the point P.
b. Q and q must be of the same sign and different magnitudes.
c. The electric field is zero at the point P.
d. The net work needed to move Q to a distance d from q was zero.
e. The net work needed to bring a test charge from infinity to the point P is zero.

I.
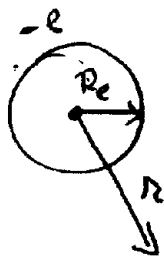
(a) For $r > R_e$ $\vec{E}$ is the same as for a point charge, by Gauss' Law & Symmetry:

$$\vec{E} = \frac{-e}{4\pi\varepsilon_0 r^2}\,\hat{r}$$
$$r > R_e$$

radially **inward**

For $r < R_e$, we can determine $\vec{E}$ from Gauss' Law. Take

$$\rho = \frac{-3e}{4\pi R_e^3}$$

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0} = \int \frac{\rho}{\varepsilon_0}\,dV$$

$$4\pi r^2 E_r = \frac{4\rho\pi r^3}{3\varepsilon_0}$$

$$E_r = \frac{\rho r}{3\varepsilon_0} = \frac{-3er}{4\pi R_e^3 \cdot 3\varepsilon_0}$$

$$\vec{E} = \frac{-er}{4\pi\varepsilon_0 R_e^3}\,\hat{r} \qquad r < R_e$$
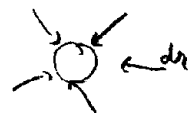
(b) For $r > R_e$, the potential is the same as for a point charge,

$$V(r) = -\int_r^\infty \vec{E}\cdot d\vec{r} = \frac{e}{4\pi\varepsilon_0}\int \frac{dr}{r^2} = \frac{-e}{4\pi\varepsilon_0 r}\Big|_r^\infty$$

$$V(r) = \frac{-e}{4\pi\varepsilon_0 r} \qquad r > R_e$$

We can determine $V(r)$ for $r < R_e$ by finding ΔV from $R_e \rightarrow r$

$$\Delta V = -\int_{R_e}^{r} \vec{E} \cdot d\vec{r}'$$

$\vec{E} \parallel d\vec{r}$, so $\vec{E} \cdot d\vec{r} = E\,dr$

$$\Delta V = \frac{e}{4\pi \varepsilon_0 R_e^3} \int_{R_e}^{r} r'\,dr' = \frac{e}{4\pi\varepsilon_0 R_e^3}\left(\frac{r'^2}{2}\right)_{r'=R_e}^{r} = \frac{e\, r^2}{8\pi\varepsilon_0 R_e^3} - \frac{e\, R_e^2}{8\pi\varepsilon_0 R_o^3}$$

$$\cdot \Delta V = \frac{e}{8\pi\varepsilon_0}\left(\frac{r^2}{R_e^3} - \frac{1}{R_e}\right)$$

Note on limits & the sign of ΔV: for $r = R_e$, ΔV must $= 0$, since we started at R_e. That is, $\Delta V = 0$ when $\Delta r = 0$. The expression for ΔV above satisfies that. As we go toward $r \rightarrow 0$, V must be decreasing, since we are moving in the same direction as $\vec{E}$ ($\vec{E} \parallel d\vec{r}$). This expression also has this behavior.
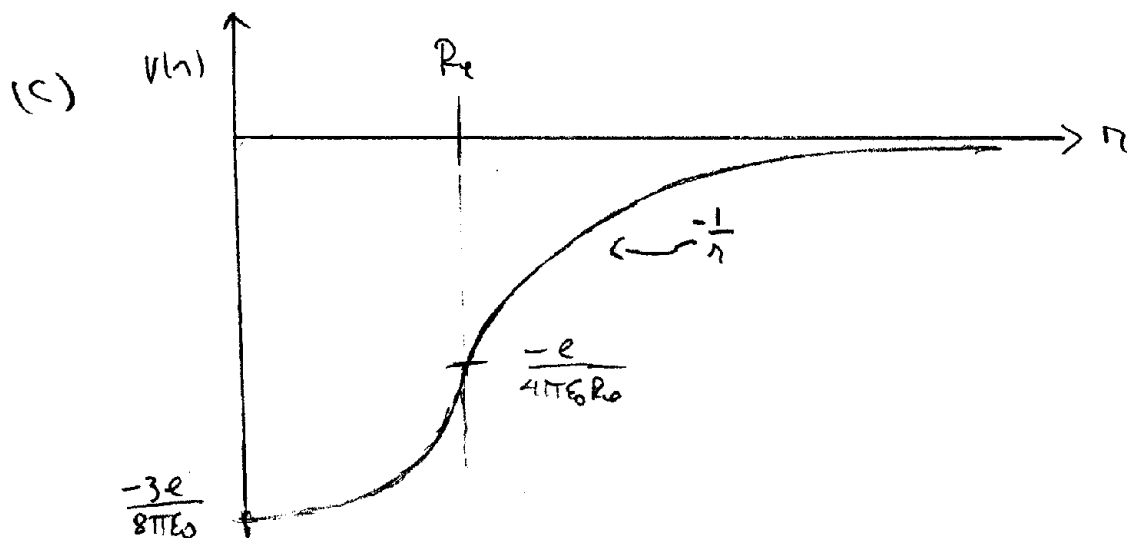
Now we must add ΔV to $V(r = R_e)$

$$V(r) = V(R_e) + \Delta V = \frac{-e}{4\pi\varepsilon_0 R_e} + \frac{e}{8\pi\varepsilon_0}\left(\frac{r^2}{R_e^3} - \frac{1}{R_e}\right)$$
$$r < R_e$$

$$\boxed{V(r) = \frac{e}{8\pi\varepsilon_0}\left(-\frac{3}{R_e} + \frac{r^2}{R_e^3}\right) \\ r < R_e}$$

(C)

(d) $u_E(r) = \dfrac{\varepsilon_0 E^2}{2}$

For $r > R_e$, $u_E(r) = \dfrac{e^2}{32\pi^2 \varepsilon_0 r^4}$

For $r < R_e$, $u_E(r) = \dfrac{e^2 r^2}{32\pi^2 \varepsilon_0 R_e^6}$

(e) Integrate $u_E(r)$ in two pieces, $U_1 \,(r > R_e)$ and $U_2 \,(r < R_e)$

$r > R_e$, $U_1 = \displaystyle\int_{r=R_e}^{\infty} u_E(r) \, dV$ $\qquad dV = 4\pi r^2 \, dr$

$U_1 = \dfrac{e^2}{32\pi^2 \varepsilon_0} \cdot 4\pi \displaystyle\int_{R_e}^{\infty} \dfrac{r^2 dr}{r^4} = \dfrac{e^2}{8\pi\varepsilon_0} \displaystyle\int_{R_e}^{\infty} \dfrac{dr}{r^2} = \dfrac{e^2}{8\pi\varepsilon_0} \left(-\dfrac{1}{r} \Big|_{R_e}^{\infty} \right)$

$U_1 = \dfrac{e^2}{8\pi\varepsilon_0 R_e}$

For $r < R_e$, $U_2 = \displaystyle\int_{0}^{R_e} u_E(r) \, dV = \dfrac{e^2 (4\pi)}{32\pi^2 \varepsilon_0 R_e^6} \displaystyle\int_{0}^{R_e} r^4 \, dr = \dfrac{e^2}{8\pi\varepsilon_0 R_e^6} \cdot \dfrac{R_e^5}{5}$

$U_2 = \dfrac{e^2}{40\pi\varepsilon_0 R_e}$

$U_{tot} = U_1 + U_2 = \dfrac{e^2}{40\pi\varepsilon_0 R_e} \left[1 + 5 \right] = \dfrac{3e^2}{20\pi\varepsilon_0 R_e}$

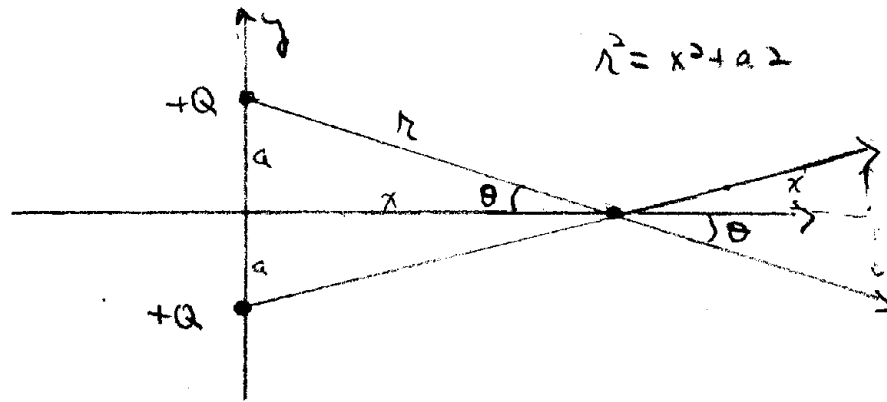$U_{tot} = \dfrac{3e^2}{20\pi\varepsilon_0 R_e}$

(f) Set $U_{tot} = m_e c^2$ (ie, energy in electric field = rest mass energy)

$$\frac{3e^2}{20\pi \varepsilon_0 R_e} = m_e c^2$$

$$\boxed{R_e = \frac{3e^2}{2\pi \varepsilon_0 \, m_e c^2}}$$

Note that numerically, $R_e \cong 2 \times 10^{-15}$ m, which is much bigger than the experimental limit $R_e < 10^{-20}$ m

II.

$r^2 = x^2 + a^2$

(a) $\vec{E}$ on the x-axis will have only an x-component from symmetry $\Rightarrow$ y-components cancel.

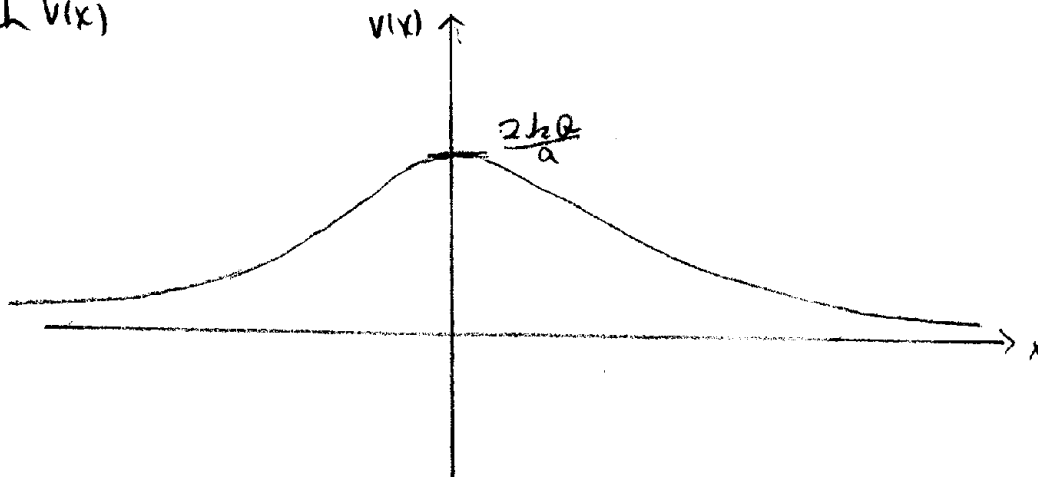$$\vec{E}(x) = E_x = \frac{2kQ}{r^2}\cos\theta \; \hat{\imath} \qquad\qquad \cos\theta = \frac{x}{r}$$

$$\boxed{\vec{E}(x) = E_x = \frac{2kQx}{(x^2+a^2)^{3/2}}\; \hat{\imath} = \frac{2Qx}{4\pi\varepsilon_0 (x^2+a^2)^{3/2}}\; \hat{\imath}}$$

for $x<0$, $\vec{E}$ changes sign, but our expression also changes sign, so it is correct for all values of x.
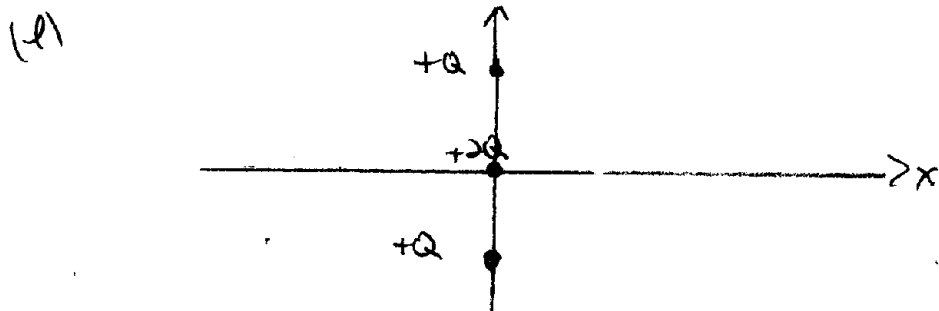
(b) V(x) is just the scalar sum of each contribution

$$\boxed{V(x) = \frac{2kQ}{r} = \frac{2kQ}{\sqrt{x^2+a^2}} = \frac{2Q}{4\pi\varepsilon_0 (x^2+a^2)^{1/2}}}$$

(c) Sketch V(x)

(d) $\vec{E}_x = 0$ where $\frac{dV}{dx} = 0$, which occurs at $x = 0$ and $x = \pm\infty$

$\boxed{\vec{E} = 0 \text{ on the x-axis at } x = 0 \ \& \ x = \pm\infty}$

(e)

$\Delta\text{work} = q\Delta V$ with $q = +2Q$

$V(x)$ at the origin is $V = \frac{2kQ}{a}$

V at $\infty = 0$, so $\Delta V = \frac{2kQ}{a}$

Therefore, the work done by an external agent to place $+2Q$ at the origin is

$$\text{Work} = \frac{2kQ}{a} \cdot 2Q = \frac{4kQ^2}{a}$$

$$\boxed{\text{Work} = \frac{4kQ^2}{a} = \frac{Q^2}{\pi\epsilon_0 a}}$$

(f) $U_{TOT} = U_{12} + U_{13} + U_{23}$

$$= \frac{kQ^2}{2a} + \frac{2kQ^2}{a} + \frac{2kQ^2}{a} = \boxed{\frac{9kQ^2}{2a} = U_{TOT}}$$

(g) If the particles are released, all of the potential energy in (f) will be converted to kinetic energy.

$$\boxed{KE = \frac{9kQ^2}{2a}}$$

How the KE is split between the three particles will depend on details of how they are released, but the sum is always the same.