[see online Class Notes 4a & 4b]

Change a Global Galerkin Approximation to a Galerkin FE Approximation (that can be automated)

Recall that the old global approximation had a set of constants (unknowns) that each were multiplied by an assumed spatial function over the entire region, that satisfied the boundary conditions.

In a Finite Element model a mesh of nodes an elements splits the region and its boundary (ies) into finite sub-regions (finite elements) whose union covers the entire region without overlaps.

$$\Omega \approx \bigcup_{e=1}^{n_e} \Omega^e \quad \text{and} \quad \Gamma = \bigcup_{b=1}^{n_b} \Gamma^b$$

region boundary

where a boundary segment is a portion (sub-set) of an element region.

$$\Gamma^b \subset \Omega^e.$$

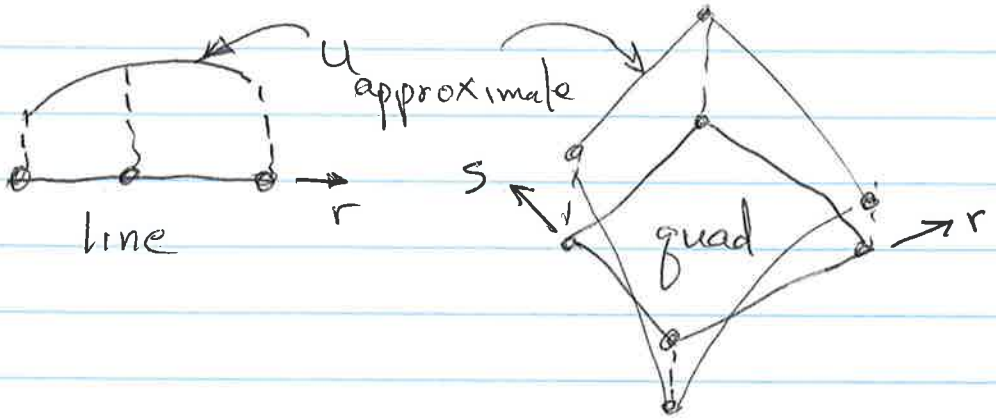
This means that the original integral over a region becomes the sum of the integrals over each element:

$$I_\Omega = \int_\Omega m \, d\Omega \approx \sum_{e=1}^{n_e} \int_{\Omega^e} m \, d\Omega$$

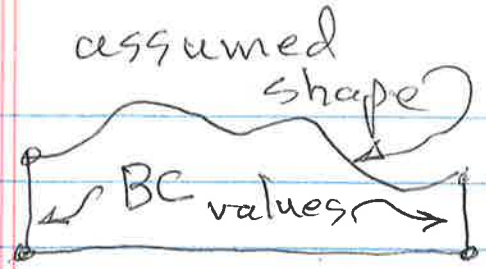
Likewise, any boundary integral becomes the sum of the integrals on the boundary segments;

$$I_\Gamma = \int_\Gamma m \, d\Gamma \approx \sum_{b=1}^{n_b} \int_{\Gamma^b} m \, d\Gamma.$$

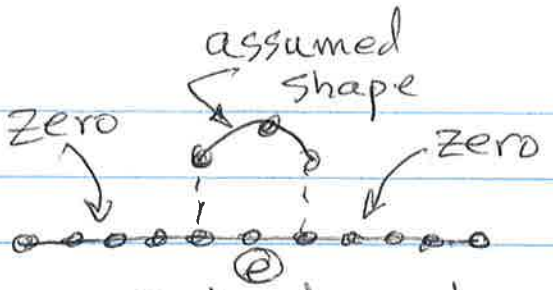
The unknowns to be found are the values of the solution at the nodes in the mesh.



Instead of multiplying each unknown by an assumed global space function each node unknown is multiplied by a polynomial space function that locally interpolates the solution within the element region using only the node values on that element.



global



finite element

The finite element spacial forms do not satisfy the boundary conditions in advance. Of course, the BC's must eventially be satisfied to obtain an unique solution.

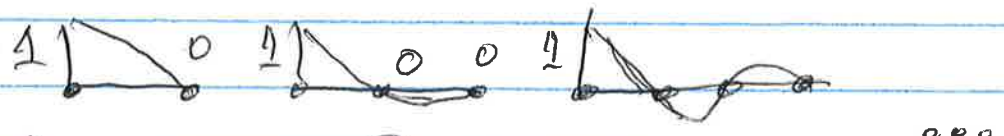
It exists as a piecewise approximation that is a polynomial inside the element, but is zero outside the element

$$u(x) \approx \sum_{e=1}^{ne} [H(x)]^e \{u^e\} = \sum_e \left(\sum_j H_j(x)\, u_j^e \right)$$

where $[H(x)]^e$ is zero outside the element. That is when

$$x \notin \Omega^e$$

Usually, the spatial form is the Lagrange interpolation polynomial. It is unity at one node and zero at all others.



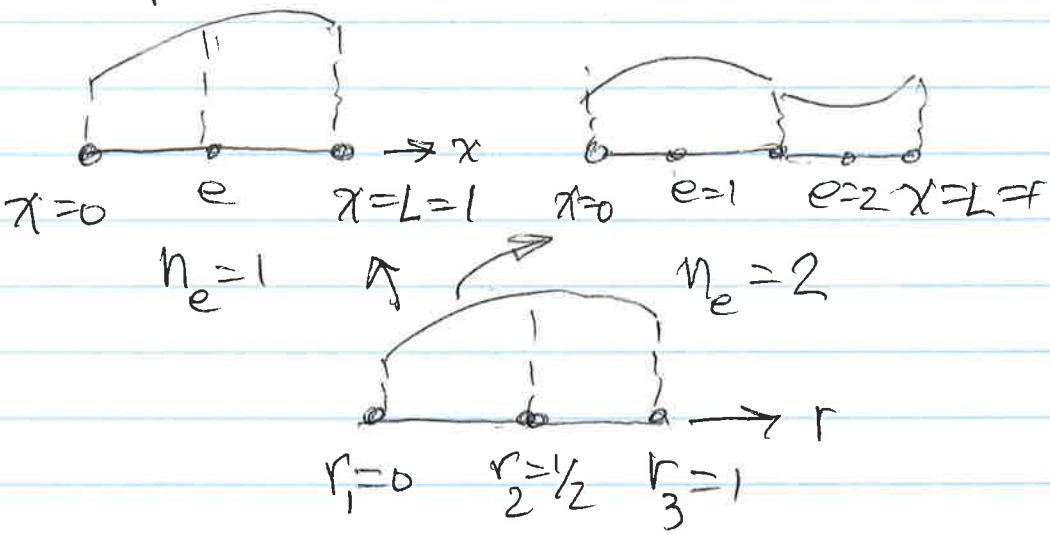
nodes 2 3 4 ...

linear quadratic cubic

Here the global Galerkin example for

$$-\frac{d^2u}{dx^2} + u + x = 0 \qquad u(0) = 0 = u(1)$$

will be repeated with one and two elements which are quadratics, whereas the prior example assumed a global cubic form



Since the approximate solution within an element is

$$u(x) = \sum_j H_j(x)\, u_j^e \quad \text{for } x \subset \Omega^e$$

$$= [H(x)]\{u^e\} = \{u^e\}^T[H(x)]$$

the residual error there is

$$R^e(x) = \sum_j \frac{d^2 H_j(x)}{dx^2} u_j^e + \sum_j H_j(x)\, u_j^e + x, \quad x \subset \Omega^e$$

$$= \left[\frac{d^2 H(x)}{dx^2}\right]\{u^e\} + [H(x)]\{u^e\} + x$$

$$= \left(\left[\frac{d^2 H(x)}{dx^2}\right] + [H(x)]\right)\{u^e\} + x$$

The Galerkin formulation requires

2.20a $$I = \int_\Omega W_i(x) R(x)\, dx = 0$$ for every i-th unknown

a mesh gives $\Omega = \bigcup_{e=1}^{n_e} \Omega^e$ and

$$\int_\Omega W_i(x) R(x)\, dx = \sum_e \int_{\Omega^e} W_i^e(x) R^e(x)\, dx = 0$$

where each element weight $W_i^e(x)$ and residual $R^e(x)$ is zero outside the element.

Inside an element the Galerkin method requires

$$W_i^e(x) \equiv H_i(x) \quad \longleftrightarrow \quad w(x) \equiv u(x)$$

In general the Galerkin method uses integration by parts to reduce the highest derivative term in $R(x)$. The online theory section at (8.1-2) shows this problem becomes

$$I = -\left[u(x)\left(1\frac{du}{dx}\right)\right]\Big|_0^L + \int_\Omega \frac{du}{dx} \cdot 1 \frac{du}{dx}\, dx + \int_\Omega u\, 1 u\, dx$$

$$+ \int_\Omega u \cdot x\, dx = 0$$

$$= -\left[\ \right]_0^L + \sum_e \int_{\Omega^e} \frac{du}{dx}\frac{du}{dx}\, dx + \sum_e \int_{\Omega^e} u \cdot u\, dx + \sum_e \int_{\Omega^e} u x\, dx = 0$$

Let the number of nodes in the mesh be n_m with each having n_g generalized unknowns, then the total number of degrees of freedom, say n_d, in the mesh is $n_d = n_m \cdot n_g$.

Likewise, the mesh contains n_e elements. If each has n_n nodes, with n_g DOF, then the number of independent degrees of freedom, on the element is $n_i = n_n \cdot n_g$.

Of course, the element unknowns are a sub-set of the system unknowns

$$\underset{n_i \times 1}{\{u^e\}} \subseteq_e \underset{n_d \times 1}{\{u\}} .$$

In detail

$$I = -\left[u\left(1\frac{du}{dx}\right)\right]_0^L + \sum_e \int_{\Omega^e} \sum_{i=1}^{n_i} \frac{dW_i^e(x)}{dx} 1 \sum_{j=1}^{n_i} \frac{dH_j(x)}{dx} dx\, u_j^e$$

$$+ \sum_e \int_{\Omega^e} \sum_{i=1}^{n_i} W_i^e(x)\, 1 \sum_{j=1}^{n_i} H_j(x)\, u_j^e\, dx$$

$$+ \sum_e \int_{\Omega^e} \sum_{i=1}^{n_u} W_i^e(x)\, x\, dx = 0$$

like theory (8.1-5)