# Classic and Good Beam Elements

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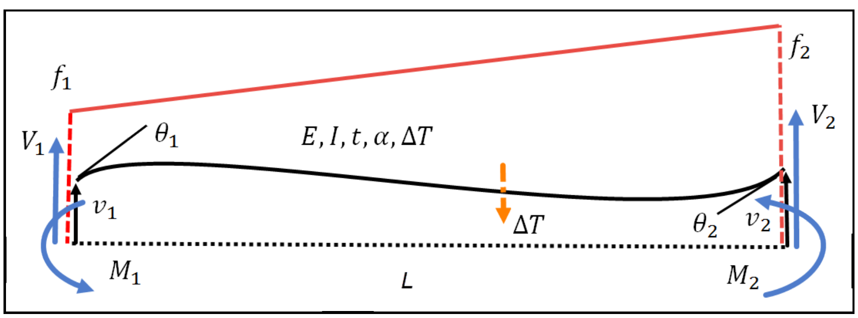
The most common fourth-order ordinary differential equation (ODE) in engineering is the Euler thin beam equation. For such a beam with a line load of *w(x)* per unit length the ODE is

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2v}{dx^2}\right] - w(x) = 0$$

where $I \equiv I_{zz} = \int_A y^2\, dA$ is the area moment of inertia of the beam cross-section and y is the direction through the depth, t, of the beam. From the theory of even order ODEs it has been shown that there are two possible essential (Dirichlet) boundary conditions: specifying the transverse displacement, v, and/or specifying the slope of the beam, $\theta \equiv dv/dx$. Since both can be applied at any point it is necessary to have both quantities as the nodal degrees of freedom in a finite element mesh ($n_g = 2$). That means that beam elements must use Hermite interpolations instead of Lagrange interpolation. The associated Neumann (non-essential) boundary conditions allow the specification of the second and/or third derivatives of the deflection ($d^2v/dx^2$ and/or $d^3v/dx^3$). In the mechanics of solids they are shown to be proportional to the bending moment and the transverse shear force, respectively, at a boundary point.

The most widely used beam element is a third degree polynomial line element with two nodes per element ($n_n = 2$) with the two generalized displacements per node being the transverse deflection *v(x)* and the rotation (slope) $\theta(x)$ at each node. Combined, the number of independent degrees of freedom on an element is $n_i = n_n * n_g = 4$, which define the complete cubic polynomial (like $c_1 + c_2x + c_3x^2 + c_4x^3$). Since the deflection and the slope (its first derivative) are continuous at the element interfaces this is known as a $C^1$ continuous element (and it is $C^\infty$ inside the element). Which is sometimes called the L2C1 element.




**Cubic beam, linear line load, point shear forces, point bending moments, and a linear temperature decrease from top to bottom through thickness t**

If a classic single two-node cubic $C^1$ beam element is in equilibrium then its matrix system (before Dirichlet BC) is

$$\frac{EI^e}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} + \frac{L}{60}\begin{bmatrix} 21 & 9 \\ 3L & 2L \\ 9 & 21 \\ -2L & -3L \end{bmatrix}\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} + \frac{\alpha^e \Delta T^e EI^e}{t^e}\begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix}$$

where L is the element length, E is the material elastic modulus, I is the cross-section moment of inertia (about an axis perpendicular to the displaced shape), $v_k$ is the transverse (in plane) displacement at node k, $\theta_k$ is the small in plane slope at node k, $V_k$ is the externally applied transverse shear force at node k (or the reaction force if $v_k$ has a specified Dirichlet value), $M_k$ is the externally applied moment (couple) at node k (or the reaction moment if slope $\theta_k$ has a specified Dirichlet value), $f_k$ is the line load (force per unit length) value at node k which is interpolated over the length of the element (using the linear L2 $C^0$ interpolation used before for linear sources), $\alpha$ is the material's coefficient of thermal expansion (CTE), and $\Delta T$ is the change in temperature from the top of beam thickness, t, to the bottom of the thickness. Note that a temperature difference causes a thermal moment, but no thermal shear force. Of course, for multiple elements in a mesh these element matrices must be assembled (scattered) to form the larger system matrix equilibrium equations; which will also be singular until they are modified to enforce the boundary conditions.

To design a beam it is necessary to know the magnitude and location of its maximum (absolute) moment and its maximum (absolute) shear force. They are given by

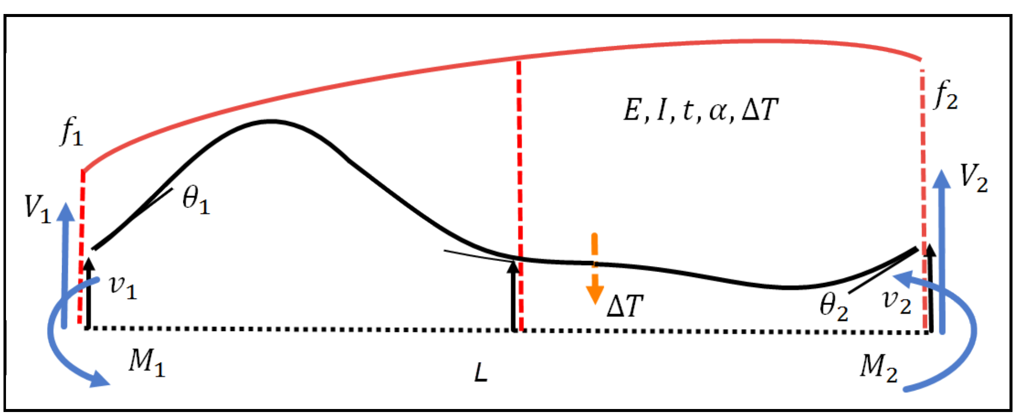
$$EI\, v''(x) = M(x) \text{ bending moment, and } EI\, v'''(x) = V(x) \text{ transverse shear force.}$$

The third derivative of this cubic beam element is a constant ($6c_4$) along the length of the beam. It is rare for a beam problem to have only segments of constant shear force. Therefore, this beam element should not be used unless you have a zillion of them in the mesh. The American Institute of Steel Construction publishes solutions for thirty nine of the most common beam support and load cases (see the appendix). A mesh of cubic beam elements can only exactly solve less than half of those common cases, unless an infinite number of elements is used.

What is a reasonable alternative to this classical beam element? The answer is to form a 5-th degree beam element by adding a third node at the center of the element that also uses the deflection and slope values ($n_n = 3, n_i = n_n * n_g = 6$ ). That element can exactly solve all 39 example AISC cases with only one to four elements as shown in the table below. Note that he figure (below) for this element shows no point force or moment at the mid-length node because either one represents a discontinuity in the post-processing results.

Number of Elements Required to Exactly Solve the AISC Cases

| Case | L3s | L2s | Case | L3s | L2s | Case | L3s | L2s |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | ∞ | 14 | 2 | 2 | 27 | 2 | ∞ |
| 2 | 1 | ∞ | 15 | 1 | ∞ | 28 | 2 | 2 |
| 3 | 2 | ∞ | 16 | 2 | 2 | 29 | 2 | ∞ |
| 4 | 3 | ∞ | 17 | 2 | 2 | 30 | 3 | 3 |
| 5 | 2 | ∞ | 18 | 1 | ∞ | 31 | 3 | 3 |
| 6 | 3 | ∞ | 19 | 1 | ∞ | 32 | 1 | ∞ |
| 7 | 2 | 2 | 20 | 1 | ∞ | 33 | 2 | 2 |
| 8 | 2 | 2 | 21 | 2 | 2 | 34 | 3 | ∞ |
| 9 | 3 | 3 | 22 | 1 | 1 | 35 | 3 | ∞ |
| 10 | 3 | 3 | 23 | 1 | 1 | 36 | 3 | ∞ |
| 11 | 3 | 3 | 24 | 2 | ∞ | 37 | 4 | ∞ |
| 12 | 1 | ∞ | 25 | 2 | ∞ | 38 | 4 | ∞ |
| 13 | 2 | 2 | 26 | 2 | ∞ | 39 | 4 | ∞ |




**Quintic beam with a quadratic line load**

If a single three-node fifth degree $C^1$ beam element is in equilibrium then its matrix system (before Dirichlet BC) is

$$\frac{EI}{35L^3}\begin{bmatrix} 5{,}092 & 1{,}138L & -3{,}584 & 1{,}920L & -1{,}508 & 242L \\ 1{,}138L & 332L^2 & -896L & 320L^2 & -242L & 38L^2 \\ -3{,}584 & -896L & 7{,}168 & 0 & -3{,}584 & 896L \\ 1{,}920L & 320L^2 & 0 & 1{,}280L^2 & -1{,}920L & 320L^2 \\ -1{,}508 & -242L & -3{,}584 & -1{,}920L & 5{,}092 & -1{,}138L \\ 242L & 38L^2 & 896L & 320L^2 & -1{,}138L & 332L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} =$$

$$= \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix} + \frac{L}{420}\begin{bmatrix} 57 & 44 & -3 \\ 3L & 4L & 0 \\ 16 & 192 & 16 \\ -8L & 0 & 8L \\ -3 & 44 & 57 \\ 0 & -4L & -3L \end{bmatrix}\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} + \frac{\alpha^e \Delta T^e EI^e}{t^e}\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{Bmatrix}$$

Keep in mind that the point shear force and the point moment (couple) both represent the Neumann boundary conditions which are weakly introduced through the integration by parts:

$$I_{NBC} = \left[v \frac{d}{dx}\left(E(x)I(x)\frac{d^2v}{dx^2}\right)\right]_0^L - \left[\frac{dv}{dx}\left(E(x)I(x)\frac{d^2v}{dx^2}\right)\right]_0^L$$

$$I_{NBC} = \left[v(L)\frac{d}{dx}\left(E(L)I(L)\frac{d^2v(L)}{dx^2}\right) + v(0)\frac{d}{dx}\left(-E(0)I(0)\frac{d^2v(0)}{dx^2}\right)\right]$$

$$+\left[\theta(L)\left(-E(L)I(L)\frac{d^2v(L)}{dx^2}\right) + \theta(0)\left(E(0)I(0)\frac{d^2v(0)}{dx^2}\right)\right]$$

$$I_{NBC} = [v(L)\,V(L) + v(0)V(0)] + \left[\theta(L)M(L) + \theta(0)M(0) = [v_2V_2 + v_1V_1] + [\theta_2\, M_2 + \theta_1\, M_1]\right]$$

$$I_{NBC} = -[v_1 \quad \theta_1 \quad v_2 \quad \theta_2]\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} \equiv -\boldsymbol{v}^{e\boldsymbol{T}}\boldsymbol{F}_{\boldsymbol{NBC}},\ \ \boldsymbol{F}_{\boldsymbol{NBC}} = \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{d}{dx}\left(E(0)I(0)\frac{d^2v(0)}{dx^2}\right) \\ E(0)I(0)\frac{d^2v(0)}{dx^2} \\ \frac{d}{dx}\left(E(L)I(L)\frac{d^2v(L)}{dx^2}\right) \\ -E(L)I(L)\frac{d^2v(L)}{dx^2} \end{Bmatrix}$$

This form is useful for applications that are not structural, but are governed by the same ordinary differential equation (like creeping fluid flow). The Hermite polynomial interpolation functions for the cubic beam are

$$v(x) = v_1(1 - 3r^2 + 2r^3) + \theta_1(r - 2r^2 + r^3)L + v_2(3r^2 - 2r^3) + \theta_2(r^3 - r^2)L$$

and the quintic functions are

$$v(r) = [v_1(1 - 23r^2 + 66r^3 - 68r^4 + 24r^5) + \theta_1(r - 6r^2 + 13r^3 - 12r^4 + 4r^5)L$$

$$+ v_2(16r^2 - 32r^3 + 16r^4) + \theta_2(-8r^2 + 32r^3 - 40r^4 + 16r^5)L$$

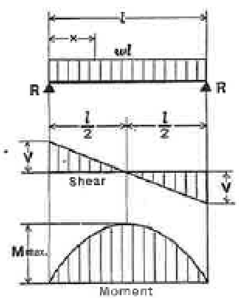
$$+ v_3(7r^2 - 34r^3 + 52r^4 - 24r^5) + \theta_3(-r^2 + 5r^3 - 8r^4 + 4r^5)L]$$

Both functions are given in the function *Hermite_1D_C1_library.m,* along with their first, second, and third derivatives *with respect* to x. They are located on the Rice Clear Linux system at /mech517/Akin_FEA_Lib.

# BEAM DIAGRAMS AND FORMULAS
## For various static loading conditions

Equivalent Tabular Load is the uniformly distributed load given in beam tables, pages 2 - 14 to 2 - 43.
For meaning of symbols, see page 2 - 118.

### 1. SIMPLE BEAM—UNIFORMLY DISTRIBUTED LOAD




Equivalent Tabular Load . . . . $= wl$

R = V . . . . . . . . . . . . $= \frac{wl}{2}$

$V_x$ . . . . . . . . . . . . $= w\left(\frac{l}{2} - x\right)$

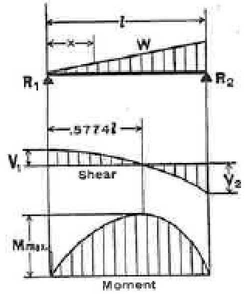
M max. (at center) . . . . . $= \frac{wl^2}{8}$

$M_x$ . . . . . . . . . . . . $= \frac{wx}{2}(l-x)$

Δmax. (at center) . . . . . $= \frac{5wl^4}{384EI}$

Δx . . . . . . . . . . . . $= \frac{wx}{24EI}(l^3 - 2lx^2 + x^3)$

### 2. SIMPLE BEAM—LOAD INCREASING UNIFORMLY TO ONE END




Equivalent Tabular Load . . . . $= \frac{16W}{9\sqrt{3}} = 1.0264W$

$R_1 = V_1$ . . . . . . . . . . . $= \frac{W}{3}$

$R_2 = V_2$ max. . . . . . . . . $= \frac{2W}{3}$

$V_x$ . . . . . . . . . . . . $= \frac{W}{3} - \frac{Wx^2}{l^2}$

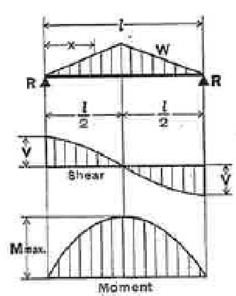
M max. $\left(\text{at } x = \frac{l}{\sqrt{3}} = .5774l\right)$ . . . $= \frac{2Wl}{9\sqrt{3}} = .1283\,Wl$

$M_x$ . . . . . . . . . . . . $= \frac{Wx}{3l^2}(l^2 - x^2)$

Δmax. $\left(\text{at } x = l\sqrt{1-\sqrt{\frac{8}{15}}} = .5193l\right)$ $= .01304\,\frac{Wl^3}{EI}$

Δx . . . . . . . . . . . . $= \frac{Wx}{180EIl^2}(3x^4 - 10l^2x^2 + 7l^4)$

### 3. SIMPLE BEAM—LOAD INCREASING UNIFORMLY TO CENTER




Equivalent Tabular Load . . . . $= \frac{4W}{3}$

R = V . . . . . . . . . . . . $= \frac{W}{2}$

$V_x$ $\left(\text{when } x < \frac{l}{2}\right)$ . . . . . $= \frac{W}{2l^2}(l^2 - 4x^2)$

M max. (at center) . . . . . $= \frac{Wl}{6}$

$M_x$ $\left(\text{when } x < \frac{l}{2}\right)$ . . . . . $= Wx\left(\frac{1}{2} - \frac{2x^2}{3l^2}\right)$

Δmax. (at center) . . . . . $= \frac{Wl^3}{60EI}$

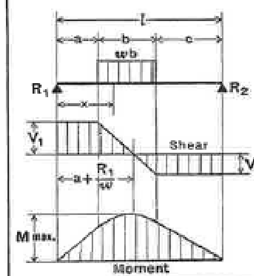
Δx . . . . . . . . . . . . $= \frac{Wx}{480EIl^2}(5l^2 - 4x^2)^2$



# BEAM DIAGRAMS AND FORMULAS
## For various static loading conditions

For meaning of symbols, see page 2 - 118.

### 4. SIMPLE BEAM—UNIFORM LOAD PARTIALLY DISTRIBUTED




$R_1 = V_1$ (max. when a < c) . . . $= \frac{wb}{2l}(2c + b)$

$R_2 = V_2$ (max. when a > c) . . . $= \frac{wb}{2l}(2a + b)$

$V_x$ (when x > a and < (a + b)) . $= R_1 - w(x - a)$

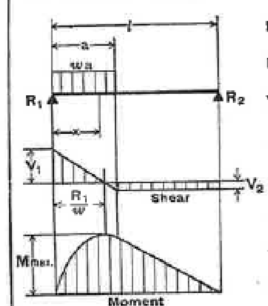
M max. $\left(\text{at } x = a + \frac{R_1}{w}\right)$ . . . . . $= R_1\left(a + \frac{R_1}{2w}\right)$

$M_x$ (when x < a) . . . . . . $= R_1 x$

$M_x$ (when x > a and < (a + b)) . $= R_1 x - \frac{w}{2}(x - a)^2$

$M_x$ (when x > (a + b)) . . . . $= R_2(l - x)$

### 5. SIMPLE BEAM—UNIFORM LOAD PARTIALLY DISTRIBUTED AT ONE END




$R_1 = V_1$ max. . . . . . . . . . $= \frac{wa}{2l}(2l - a)$

$R_2 = V_2$ . . . . . . . . . . . $= \frac{wa^2}{2l}$

V (when x < a) . . . . . . $= R_1 - wx$

M max. $\left(\text{at } x = \frac{R_1}{w}\right)$ . . . . . . $= \frac{R_1^2}{2w}$

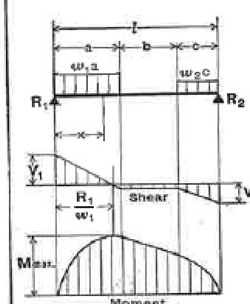
$M_x$ (when x < a) . . . . . . $= R_1 x - \frac{wx^2}{2}$

$M_x$ (when x > a) . . . . . . $= R_2(l - x)$

Δx (when x < a) . . . . . . $= \frac{wx}{24EIl}\left(a^2(2l-a)^2 - 2ax^2(2l-a) + lx^3\right)$

Δx (when x > a) . . . . . . $= \frac{wa^2(l-x)}{24EIl}(4xl - 2x^2 - a^2)$

### 6. SIMPLE BEAM—UNIFORM LOAD PARTIALLY DISTRIBUTED AT EACH END




$R_1 = V_1$ . . . . . . . . . . . $= \frac{w_1 a(2l - a) + w_2 c^2}{2l}$

$R_2 = V_2$ . . . . . . . . . . . $= \frac{w_2 c(2l - c) + w_1 a^2}{2l}$

$V_x$ (when x < a) . . . . . . $= R_1 - w_1 x$

$V_x$ (when x > a and < (a + b)) . $= R_1 - R_2$

$V_x$ (when x > (a + b)) . . . . $= R_2 - w_2(l - x)$

M max. $\left(\text{at } x = \frac{R_1}{w_1} \text{ when } R_1 < w_1 a\right)$ . $= \frac{R_1^2}{2w_1}$

M max. $\left(\text{at } x = l - \frac{R_2}{w_2} \text{ when } R_2 < w_2 c\right)$ $= \frac{R_2^2}{2w_2}$

$M_x$ (when x < a) . . . . . . $= R_1 x - \frac{w_1 x^2}{2}$

$M_x$ (when x > a and < (a + b)) . $= R_1 x - \frac{w_1 a}{2}(2x - a)$

$M_x$ (when x > (a + b)) . . . . $= R_2(l - x) - \frac{w_2(l - x)^2}{2}$

# BEAM DIAGRAMS AND FORMULAS

## For various static loading conditions

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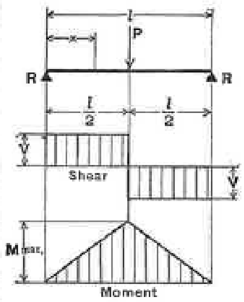
### 7. SIMPLE BEAM—CONCENTRATED LOAD AT CENTER




Equivalent Tabular Load . . . . . . $= 2P$

$R = V$ . . . . . . . . . . . . $= \frac{P}{2}$

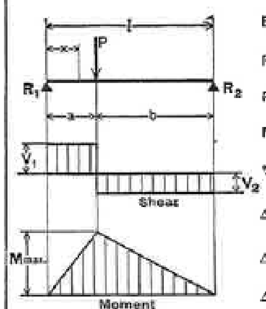
M max. (at point of load) . . . . . . $= \frac{Pl}{4}$

$M_x$ (when $x < \frac{l}{2}$) . . . . . . $= \frac{Px}{2}$

$\Delta$max. (at point of load) . . . . . . $= \frac{Pl^3}{48EI}$

$\Delta x$ (when $x < \frac{l}{2}$) . . . . . . $= \frac{Px}{48EI}(3l^2 - 4x^2)$

### 8. SIMPLE BEAM—CONCENTRATED LOAD AT ANY POINT




Equivalent Tabular Load . . . . . . $= \frac{8\,Pab}{l^2}$

$R_1 = V_1$ (max. when $a < b$) . . . . $= \frac{Pb}{l}$

$R_2 = V_2$ (max. when $a > b$) . . . . $= \frac{Pa}{l}$

M max. (at point of load) . . . . $= \frac{Pab}{l}$

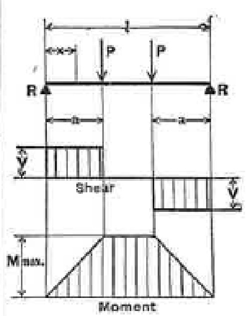
$M_x$ (when $x < a$) . . . . . $= \frac{Pbx}{l}$

$\Delta$max. $\left(\text{at } x = \sqrt{\frac{a(a+2b)}{3}} \text{ when } a > b\right)$ $= \frac{Pab\,(a+2b)\,\sqrt{3a\,(a+2b)}}{27\,EI\,l}$

$\Delta a$ (at point of load) . . . . . $= \frac{Pa^2b^2}{3EI\,l}$

$\Delta x$ (when $x < a$) . . . . . $= \frac{Pbx}{6EI\,l}(l^2 - b^2 - x^2)$

### 9. SIMPLE BEAM—TWO EQUAL CONCENTRATED LOADS SYMMETRICALLY PLACED




Equivalent Tabular Load . . . . . . $= \frac{8\,Pa}{l}$

$R = V$ . . . . . . . . . . . . $= P$

M max. (between loads) . . . . . . $= Pa$

$M_x$ (when $x < a$) . . . . . . $= Px$

$\Delta$max. (at center) . . . . . . . . $= \frac{Pa}{24EI}(3l^2 - 4a^2)$

$\Delta x$ (when $x < a$) . . . . . . . $= \frac{Px}{6EI}(3la - 3a^2 - x^2)$

$\Delta x$ (when $x > a$ and $< (l - a)$) . . $= \frac{Pa}{6EI}(3lx - 3x^2 - a^2)$



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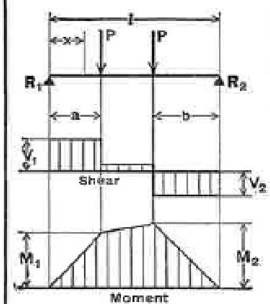
### 10. SIMPLE BEAM—TWO EQUAL CONCENTRATED LOADS UNSYMMETRICALLY PLACED




$R_1 = V_1$ (max. when $a < b$) . . . . . $= \frac{P}{l}(l - a + b)$

$R_2 = V_2$ (max. when $a > b$) . . . . . $= \frac{P}{l}(l - b + a)$

$V_x$ (when $x > a$ and $< (l - b)$) . . $= \frac{P}{l}(b - a)$

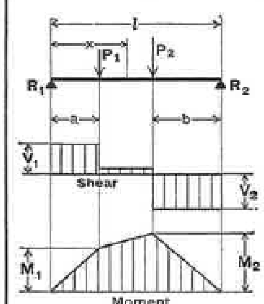
$M_1$ (max. when $a > b$) . . . . . $= R_1a$

$M_2$ (max. when $a < b$) . . . . . $= R_2b$

$M_x$ (when $x < a$) . . . . . . . $= R_1x$

$M_x$ (when $x > a$ and $< (l - b)$) . . $= R_1x - P(x - a)$

### 11. SIMPLE BEAM—TWO UNEQUAL CONCENTRATED LOADS UNSYMMETRICALLY PLACED




$R_1 = V_1$ . . . . . . . . . . . . . $= \frac{P_1(l - a) + P_2b}{l}$

$R_2 = V_2$ . . . . . . . . . . . . . $= \frac{P_1a + P_2(l - b)}{l}$

$V_x$ (when $x > a$ and $< (l - b)$) . . $= R_1 - P_1$

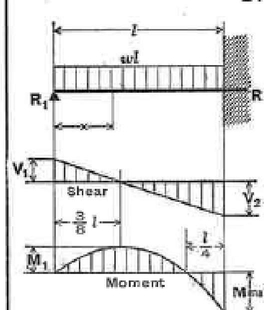
$M_1$ (max. when $R_1 < P_1$) . . . . $= R_1a$

$M_2$ (max. when $R_2 < P_2$) . . . . $= R_2b$

$M_x$ (when $x < a$) . . . . . . . $= R_1x$

$M_x$ (when $x > a$ and $< (l - b)$) . . $= R_1x - P_1(x - a)$

### 12. BEAM FIXED AT ONE END, SUPPORTED AT OTHER—UNIFORMLY DISTRIBUTED LOAD




Equivalent Tabular Load . . . . . . $= wl$

$R_1 = V_1$ . . . . . . . . . . . . . $= \frac{3wl}{8}$

$R_2 = V_2$ max. . . . . . . . . . . $= \frac{5wl}{8}$

$V_x$ . . . . . . . . . . . . . . $= R_1 - wx$

M max. . . . . . . . . . . . . . $= \frac{wl^2}{8}$

$M_1$ $\left(\text{at } x = \frac{3}{8}l\right)$ . . . . . . . . $= \frac{9}{128}wl^2$

$M_x$ . . . . . . . . . . . . . . $= R_1x - \frac{wx^2}{2}$

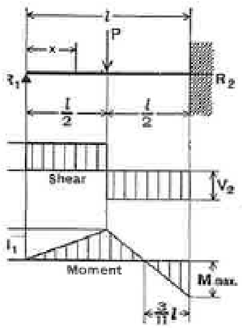
$\Delta$max. $\left(\text{at } x = \frac{l}{16}(1 + \sqrt{33}) = .4215l\right)$ . $= \frac{wl^4}{185EI}$

$\Delta x$ . . . . . . . . . . . . . . $= \frac{wx}{48EI}(l^3 - 3lx^2 + 2x^3)$

# BEAM DIAGRAMS AND FORMULAS
## For various static loading conditions

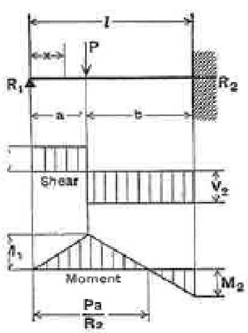
quivalent Tabular Load is the uniformly distributed load given in beam tables, pages 2-14 to 2-43. For meaning of symbols, see page 2-118.

### 13. BEAM FIXED AT ONE END, SUPPORTED AT OTHER—CONCENTRATED LOAD AT CENTER




| | | |
|---|---|---|
| Equivalent Tabular Load | | $= \frac{3P}{2}$ |
| $R_1 = V_1$ | | $= \frac{5P}{16}$ |
| $R_2 = V_2$ max. | | $= \frac{11P}{16}$ |
| M max. | (at fixed end) | $= \frac{3Pl}{16}$ |
| $M_1$ | (at point of load) | $= \frac{5Pl}{32}$ |
| $M_x$ | $\left(\text{when } x < \frac{l}{2}\right)$ | $= \frac{5Px}{16}$ |
| $M_x$ | $\left(\text{when } x > \frac{l}{2}\right)$ | $= P\left(\frac{l}{2} - \frac{11x}{16}\right)$ |
| $\Delta$max. | $\left(\text{at } x = l\sqrt{\frac{1}{5}} = .4472l\right)$ | $= \frac{Pl^3}{48EI\sqrt{5}} = .009317\frac{Pl^3}{EI}$ |
| $\Delta x$ | (at point of load) | $= \frac{7Pl^3}{768EI}$ |
| $\Delta x$ | $\left(\text{when } x < \frac{l}{2}\right)$ | $= \frac{Px}{96EI}(3l^2 - 5x^2)$ |
| $\Delta x$ | $\left(\text{when } x > \frac{l}{2}\right)$ | $= \frac{P}{96EI}(x - l)^2(11x - 2l)$ |

### 14. BEAM FIXED AT ONE END, SUPPORTED AT OTHER—CONCENTRATED LOAD AT ANY POINT



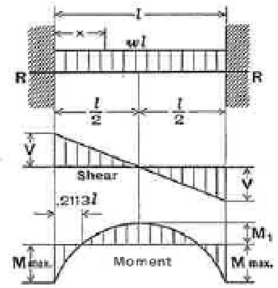

| | | |
|---|---|---|
| $R_1 = V_1$ | | $= \frac{Pb^2}{2l^3}(a + 2l)$ |
| $R_2 = V_2$ | | $= \frac{Pa}{2l^3}(3l^2 - a^2)$ |
| $M_1$ | (at point of load) | $= R_1 a$ |
| $M_2$ | (at fixed end) | $= \frac{Pab}{2l^2}(a + l)$ |
| $M_x$ | (when $x < a$) | $= R_1 x$ |
| $M_x$ | (when $x > a$) | $= R_1 x - P(x - a)$ |
| $\Delta$max. | $\left(\text{when } a < .414l \text{ at } x = l\frac{l^2+a^2}{3l^2-a^2}\right)$ | $= \frac{Pa}{3EI}\frac{(l^2 - a^2)^3}{(3l^2 - a^2)^2}$ |
| $\Delta$max. | $\left(\text{when } a > .414l \text{ at } x = l\sqrt{\frac{a}{2l+a}}\right)$ | $= \frac{Pab^2}{6EI}\sqrt{\frac{a}{2l + a}}$ |
| $\Delta a$ | (at point of load) | $= \frac{Pa^2b^3}{12EIl^3}(3l + a)$ |
| $\Delta x$ | (when $x < a$) | $= \frac{Pb^2x}{12EIl^3}(3al^2 - 2lx^2 - ax^2)$ |
| $\Delta x$ | (when $x > a$) | $= \frac{Pa}{12EIl^3}(l-x)^2(3l^2x - a^2x - 2a^2l)$ |



# BEAM DIAGRAMS AND FORMULAS
## For various static loading conditions

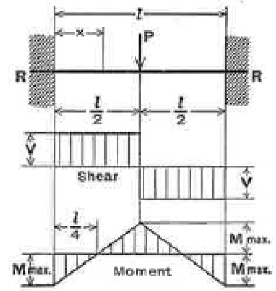
Equivalent Tabular Load is the uniformly distributed load given in beam tables, pages 2-14 to 2-43. For meaning of symbols, see page 2-118.

### 15. BEAM FIXED AT BOTH ENDS—UNIFORMLY DISTRIBUTED LOADS



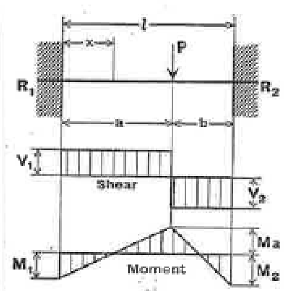

| | | |
|---|---|---|
| Equivalent Tabular Load | | $= \frac{2wl}{3}$ |
| $R = V$ | | $= \frac{wl}{2}$ |
| $V_x$ | | $= w\left(\frac{l}{2} - x\right)$ |
| M max. | (at ends) | $= \frac{wl^2}{12}$ |
| $M_1$ | (at center) | $= \frac{wl^2}{24}$ |
| $M_x$ | | $= \frac{w}{12}(6lx - l^2 - 6x^2)$ |
| $\Delta$max. | (at center) | $= \frac{wl^4}{384EI}$ |
| $\Delta x$ | | $= \frac{wx^2}{24EI}(l - x)^2$ |

### 16. BEAM FIXED AT BOTH ENDS—CONCENTRATED LOAD AT CENTER




| | | |
|---|---|---|
| Equivalent Tabular Load | | $= P$ |
| $R = V$ | | $= \frac{P}{2}$ |
| M max. | (at center and ends) | $= \frac{Pl}{8}$ |
| $M_x$ | $\left(\text{when } x < \frac{l}{2}\right)$ | $= \frac{P}{8}(4x - l)$ |
| $\Delta$max. | (at center) | $= \frac{Pl^3}{192EI}$ |
| $\Delta x$ | | $= \frac{Px^2}{48EI}(3l - 4x)$ |

### 17. BEAM FIXED AT BOTH ENDS—CONCENTRATED LOAD AT ANY POINT




| | | |
|---|---|---|
| $R_1 = V_1$ | (max. when $a < b$) | $= \frac{Pb^2}{l^3}(3a + b)$ |
| $R_2 = V_2$ | (max. when $a > b$) | $= \frac{Pa^2}{l^3}(a + 3b)$ |
| $M_1$ | (max. when $a < b$) | $= \frac{Pab^2}{l^2}$ |
| $M_2$ | (max. when $a > b$) | $= \frac{Pa^2b}{l^2}$ |
| $M_a$ | (at point of load) | $= \frac{2Pa^2b^2}{l^3}$ |
| $M_x$ | (when $x < a$) | $= R_1 x - \frac{Pab^2}{l^2}$ |
| $\Delta$max. | $\left(\text{when } a > b \text{ at } x = \frac{2al}{3a + b}\right)$ | $= \frac{2Pa^3b^2}{3EI(3a + b)^2}$ |
| $\Delta a$ | (at point of load) | $= \frac{Pa^3b^3}{3EIl^3}$ |
| $\Delta x$ | (when $x < a$) | $= \frac{Pb^2x^2}{6EIl^3}(3al - 3ax - bx)$ |

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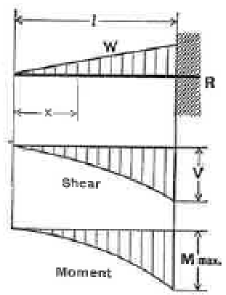
# BEAM DIAGRAMS AND FORMULAS

## For various static loading conditions

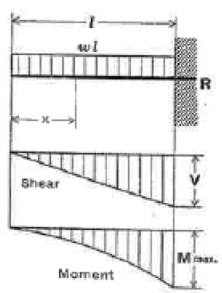
Equivalent Tabular Load is the uniformly distributed load given in beam tables, pages 2 - 14 to 2 - 43. For meaning of symbols, see page 2 - 118.

### 18. CANTILEVER BEAM—LOAD INCREASING UNIFORMLY TO FIXED END



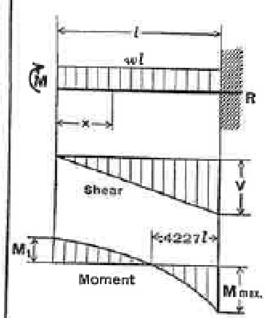

| | |
|---|---|
| Equivalent Tabular Load | $= \frac{8}{3} W$ |
| R = V | $= W$ |
| $V_x$ | $= W \frac{x^2}{l^2}$ |
| M max. (at fixed end) | $= \frac{Wl}{3}$ |
| $M_x$ | $= \frac{Wx^3}{3l^2}$ |
| $\Delta$max. (at free end) | $= \frac{Wl^3}{15EI}$ |
| $\Delta x$ | $= \frac{W}{60EIl^2}(x^5 - 5l^4x + 4l^5)$ |

### 19. CANTILEVER BEAM—UNIFORMLY DISTRIBUTED LOAD




| | |
|---|---|
| Equivalent Tabular Load | $= 4wl$ |
| R = V | $= wl$ |
| $V_x$ | $= wx$ |
| M max. (at fixed end) | $= \frac{wl^2}{2}$ |
| $M_x$ | $= \frac{wx^2}{2}$ |
| $\Delta$max. (at free end) | $= \frac{wl^4}{8EI}$ |
| $\Delta x$ | $= \frac{w}{24EI}(x^4 - 4l^3x + 3l^4)$ |

### 20. BEAM FIXED AT ONE END, FREE TO DEFLECT VERTICALLY BUT NOT ROTATE AT OTHER—UNIFORMLY DISTRIBUTED LOAD




The deflection at the guided end is assumed to be in a vertical plane.

| | |
|---|---|
| Equivalent Tabular Load | $= \frac{8}{3} wl$ |
| R = V | $= wl$ |
| $V_x$ | $= wx$ |
| M max. (at fixed end) | $= \frac{wl^2}{3}$ |
| $M_1$ (at deflected end) | $= \frac{wl^2}{6}$ |
| $M_x$ | $= \frac{w}{6}(l^2 - 3x^2)$ |
| $\Delta$max. (at deflected end) | $= \frac{wl^4}{24EI}$ |
| $\Delta x$ | $= \frac{w(l^2 - x^2)^2}{24EI}$ |

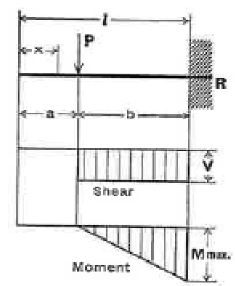


# BEAM DIAGRAMS AND FORMULAS

## For various static loading conditions

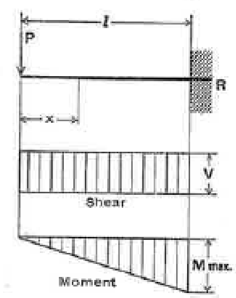
Equivalent Tabular Load is the uniformly distributed load given in beam tables, pages 2 - 14 to 2 - 43. For meaning of symbols, see page 2 - 118.

### 21. CANTILEVER BEAM—CONCENTRATED LOAD AT ANY POINT



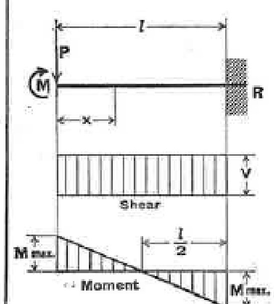

| | |
|---|---|
| Equivalent Tabular Load | $= \frac{8Pb}{l}$ |
| R = V (when x < a) | $= P$ |
| M max. (at fixed end) | $= Pb$ |
| $M_x$ (when x > a) | $= P(x - a)$ |
| $\Delta$max. (at free end) | $= \frac{Pb^2}{6EI}(3l - b)$ |
| $\Delta a$ (at point of load) | $= \frac{Pb^3}{3EI}$ |
| $\Delta x$ (when x < a) | $= \frac{Pb^2}{6EI}(3l - 3x - b)$ |
| $\Delta x$ (when x > a) | $= \frac{P(l - x)^2}{6EI}(3b - l + x)$ |

### 22. CANTILEVER BEAM—CONCENTRATED LOAD AT FREE END




| | |
|---|---|
| Equivalent Tabular Load | $= 8P$ |
| R = V | $= P$ |
| M max. (at fixed end) | $= Pl$ |
| $M_x$ | $= Px$ |
| $\Delta$max. (at free end) | $= \frac{Pl^3}{3EI}$ |
| $\Delta x$ | $= \frac{P}{6EI}(2l^3 - 3l^2x + x^3)$ |

### 23. BEAM FIXED AT ONE END, FREE TO DEFLECT VERTICALLY BUT NOT ROTATE AT OTHER—CONCENTRATED LOAD AT DEFLECTED END




The deflection at the guided end is assumed to be in a vertical plane.

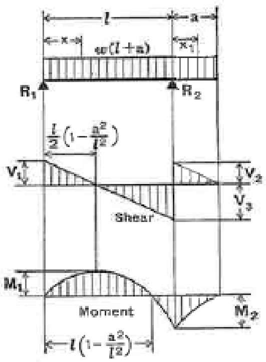
| | |
|---|---|
| Equivalent Tabular Load | $= 4P$ |
| R = V | $= P$ |
| M max. (at both ends) | $= \frac{Pl}{2}$ |
| $M_x$ | $= P\left(\frac{l}{2} - x\right)$ |
| $\Delta$max. (at deflected end) | $= \frac{Pl^3}{12EI}$ |
| $\Delta x$ | $= \frac{P(l - x)^2}{12EI}(l + 2x)$ |

# BEAM DIAGRAMS AND FORMULAS

## For various static loading conditions

For meaning of symbols, see page 2 - 118.

### 24. BEAM OVERHANGING ONE SUPPORT—UNIFORMLY DISTRIBUTED LOAD




$R_1 = V_1$ . . . . . . . . . . $= \frac{w}{2l}(l^2 - a^2)$

$R_2 = V_2 + V_3$ . . . . . . . . $= \frac{w}{2l}(l + a)^2$

$V_2$ . . . . . . . . . . . . $= wa$

$V_3$ . . . . . . . . . . . . $= \frac{w}{2l}(l^2 + a^2)$

$V_x$ (between supports) . . . $= R_1 - wx$

$V_{x_1}$ (for overhang) . . . . $= w(a - x_1)$

$M_1$ $\left(\text{at } x = \frac{l}{2}\left[1 - \frac{a^2}{l^2}\right]\right)$ . . . $= \frac{w}{8l^2}(l + a)^2(l - a)^2$

$M_2$ (at $R_2$) . . . . . . . $= \frac{wa^2}{2}$

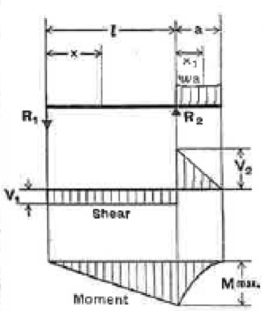
$M_x$ (between supports) . . . $= \frac{wx}{2l}(l^2 - a^2 - xl)$

$M_{x_1}$ (for overhang) . . . . $= \frac{w}{2}(a - x_1)^2$

$\Delta x$ (between supports) . . . $= \frac{wx}{24EIl}(l^4 - 2l^2x^2 + lx^3 - 2a^2l^2 + 2a^2x^2)$

$\Delta x_1$ (for overhang) . . . . $= \frac{wx_1}{24EI}(4a^2l - l^3 + 6a^2x_1 - 4ax_1^2 + x_1^3)$

### 25. BEAM OVERHANGING ONE SUPPORT—UNIFORMLY DISTRIBUTED LOAD ON OVERHANG




$R_1 = V_1$ . . . . . . . . . . $= \frac{wa^2}{2l}$

$R_2 = V_1 + V_2$ . . . . . . . . $= \frac{wa}{2l}(2l + a)$

$V_2$ . . . . . . . . . . . . $= wa$

$V_{x_1}$ (for overhang) . . . . . $= w(a - x_1)$

M max. (at $R_2$) . . . . . . $= \frac{wa^2}{2}$

$M_x$ (between supports) . . . $= \frac{wa^2x}{2l}$

$M_{x_1}$ (for overhang) . . . . . $= \frac{w}{2}(a - x_1)^2$

$\Delta$max. $\left(\text{between supports at } x = \frac{l}{\sqrt{3}}\right) = \frac{wa^2l^2}{18\sqrt{3}EI} = .03208\frac{wa^2l^2}{EI}$

$\Delta$max. (for overhang at $x_1 = a$) . $= \frac{wa^3}{24EI}(4l + 3a)$

$\Delta x$ (between supports) . . . $= \frac{wa^2x}{12EIl}(l^2 - x^2)$

$\Delta x_1$ (for overhang) . . . . $= \frac{wx_1}{24EI}(4a^2l + 6a^2x_1 - 4ax_1^2 + x_1^3)$

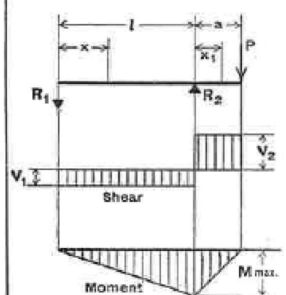


# BEAM DIAGRAMS AND FORMULAS

## For various static loading conditions

Equivalent Tabular Load is the uniformly distributed load given in beam tables, pages 2 - 14 to 2 - 43.
For meaning of symbols, see page 2 - 118.

### 26. BEAM OVERHANGING ONE SUPPORT—CONCENTRATED LOAD AT END OF OVERHANG




$R_1 = V_1$ . . . . . . . . . . $= \frac{Pa}{l}$

$R_2 = V_1 + V_2$ . . . . . . . . $= \frac{P}{l}(l + a)$

$V_2$ . . . . . . . . . . . . $= P$

M max. (at $R_2$) . . . . . . . $= Pa$

$M_x$ (between supports) . . . $= \frac{Pax}{l}$

$M_{x_1}$ (for overhang) . . . . $= P(a - x_1)$

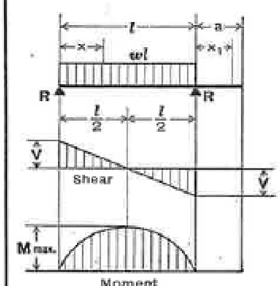
$\Delta$max. $\left(\text{between supports at } x = \frac{l}{\sqrt{3}}\right) = \frac{Pal^2}{9\sqrt{3}EI} = .06415\frac{Pal^2}{EI}$

$\Delta$max. (for overhang at $x_1 = a$) . $= \frac{Pa^2}{3EI}(l + a)$

$\Delta x$ (between supports) . . . $= \frac{Pax}{6EIl}(l^2 - x^2)$

$\Delta x_1$ (for overhang) . . . . $= \frac{Px_1}{6EI}(2al + 3ax_1 - x_1^2)$

### 27. BEAM OVERHANGING ONE SUPPORT—UNIFORMLY DISTRIBUTED LOAD BETWEEN SUPPORTS




Equivalent Tabular Load . . . . $= wl$

$R = V$ . . . . . . . . . . . $= \frac{wl}{2}$

$V_x$ . . . . . . . . . . . . $= w\left(\frac{l}{2} - x\right)$

M max. (at center) . . . . . . $= \frac{wl^2}{8}$

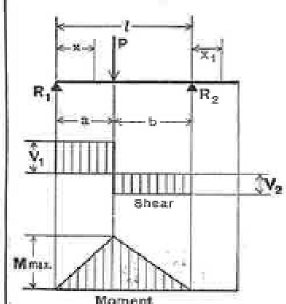
$M_x$ . . . . . . . . . . . . $= \frac{wx}{2}(l - x)$

$\Delta$max. (at center) . . . . . . $= \frac{5wl^4}{384EI}$

$\Delta x$ . . . . . . . . . . . . $= \frac{wx}{24EI}(l^3 - 2lx^2 + x^3)$

$\Delta x_1$ . . . . . . . . . . . $= \frac{wl^3x_1}{24EI}$

### 28. BEAM OVERHANGING ONE SUPPORT—CONCENTRATED LOAD AT ANY POINT BETWEEN SUPPORTS




Equivalent Tabular Load . . . . $= \frac{8Pab}{l^2}$

$R_1 = V_1$ (max. when $a < b$) . . . $= \frac{Pb}{l}$

$R_2 = V_2$ (max. when $a > b$) . . . $= \frac{Pa}{l}$

M max. (at point of load) . . . $= \frac{Pab}{l}$

$M_x$ (when $x < a$) . . . . . $= \frac{Pbx}{l}$

$\Delta$max. $\left(\text{at } x = \sqrt{\frac{a(a+2b)}{3}} \text{ when } a > b\right) = \frac{Pab\,(a + 2b)\,\sqrt{3a\,(a + 2b)}}{27EIl}$

$\Delta a$ (at point of load) . . . $= \frac{Pa^2b^2}{3EIl}$

$\Delta x$ (when $x < a$) . . . . . $= \frac{Pbx}{6EIl}(l^2 - b^2 - x^2)$

$\Delta x$ (when $x > a$) . . . . . $= \frac{Pa\,(l - x)}{6EIl}(2lx - x^2 - a^2)$

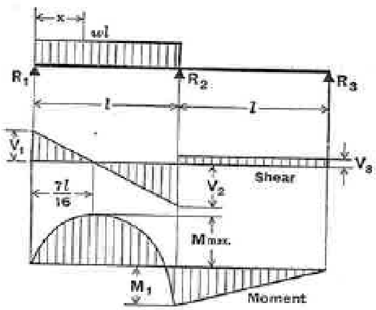
$\Delta x_1$ . . . . . . . . . . . $= \frac{Pabx_1}{6EIl}(l + a)$

# BEAM DIAGRAMS AND FORMULAS

## For various static loading conditions

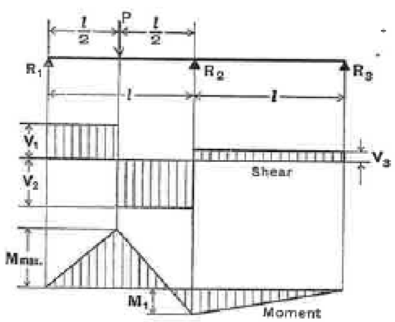
Equivalent Tabular Load is the uniformly distributed load given in beam tables, pages 2-14 to 2-43.
For meaning of symbols, see page 2-118.

### 29. CONTINUOUS BEAM—TWO EQUAL SPANS—UNIFORM LOAD ON ONE SPAN



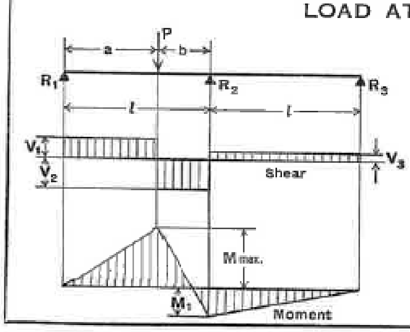
| | |
|---|---|
| Equivalent Tabular Load | $= \frac{49}{64} wl$ |
| $R_1 = V_1$ | $= \frac{7}{16} wl$ |
| $R_2 = V_2 + V_3$ | $= \frac{5}{8} wl$ |
| $R_3 = V_3$ | $= -\frac{1}{16} wl$ |
| $V_2$ | $= \frac{9}{16} wl$ |
| M Max. $\left(\text{at } x = \frac{7}{16} l\right)$ | $= \frac{49}{512} wl^2$ |
| $M_1$ (at support $R_2$) | $= \frac{1}{16} wl^2$ |
| $M_x$ (when $x < l$) | $= \frac{wx}{16}(7l - 8x)$ |
| $\Delta$ Max. (0.472 $l$ from $R_1$) | $= 0.0092\ wl^4/EI$ |

### 30. CONTINUOUS BEAM—TWO EQUAL SPANS—CONCENTRATED LOAD AT CENTER OF ONE SPAN



| | |
|---|---|
| Equivalent Tabular Load | $= \frac{13}{8} P$ |
| $R_1 = V_1$ | $= \frac{13}{32} P$ |
| $R_2 = V_2 + V_3$ | $= \frac{11}{16} P$ |
| $R_3 = V_3$ | $= -\frac{3}{32} P$ |
| $V_2$ | $= \frac{19}{32} P$ |
| M Max. (at point of load) | $= \frac{13}{64} Pl$ |
| $M_1$ (at support $R_2$) | $= \frac{3}{32} Pl$ |
| $\Delta$ Max. (0.480 $l$ from $R_1$) | $= 0.015\ Pl^3/EI$ |

### 31. CONTINUOUS BEAM—TWO EQUAL SPANS—CONCENTRATED LOAD AT ANY POINT



| | |
|---|---|
| $R_1 = V_1$ | $= \frac{Pb}{4l^3}\left(4l^2 - a(l+a)\right)$ |
| $R_2 = V_2 + V_3$ | $= \frac{Pa}{2l^3}\left(2l^2 + b(l+a)\right)$ |
| $R_3 = V_3$ | $= -\frac{Pab}{4l^3}(l+a)$ |
| $V_2$ | $= \frac{Pa}{4l^3}\left(4l^2 + b(l+a)\right)$ |
| M max. (at point of load) | $= \frac{Pab}{4l^3}\left(4l^2 - a(l+a)\right)$ |
| $M_1$ (at support $R_2$) | $= \frac{Pab}{4l^2}(l+a)$ |

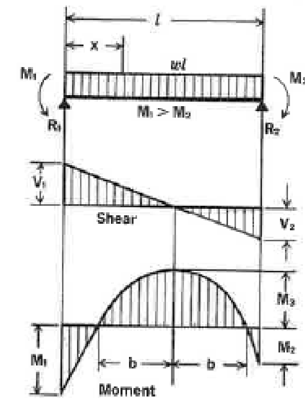


# BEAM DIAGRAMS AND FORMULAS

## For various static loading conditions

For meaning of symbols, see page 2-118

### 32. BEAM—UNIFORMLY DISTRIBUTED LOAD AND VARIABLE END MOMENTS



$$R_1 = V_1 = \frac{wl}{2} + \frac{M_1 - M_2}{l}$$

$$R_2 = V_2 = \frac{wl}{2} - \frac{M_1 - M_2}{l}$$

$$V_x = w\left(\frac{l}{2} - x\right) + \frac{M_1 - M_2}{l}$$

$$M_3 \left(\text{at } x = \frac{l}{2} + \frac{M_1 - M_2}{wl}\right) = \frac{wl^2}{8} - \frac{M_1 + M_2}{2} + \frac{(M_1 - M_2)^2}{2wl^2}$$

$$M_x = \frac{wx}{2}(l - x) + \left(\frac{M_1 - M_2}{l}\right)x - M_1$$

$$b\left(\begin{array}{l}\text{To locate}\\ \text{inflection points}\end{array}\right) = \sqrt{\frac{l^2}{4} - \left(\frac{M_1 + M_2}{w}\right) + \left(\frac{M_1 - M_2}{wl}\right)^2}$$

$$\Delta_x = \frac{wx}{24EI}\left[x^3 - \left(2l - \frac{4M_1}{wl} + \frac{4M_2}{wl}\right)x^2 + \frac{12M_1}{w}x + l^3 - \frac{8M_1 l}{w} - \frac{4M_2 l}{w}\right]$$

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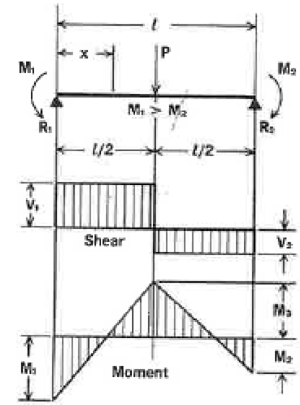
### 33. BEAM—CONCENTRATED LOAD AT CENTER AND VARIABLE END MOMENTS



$$R_1 = V_1 = \frac{P}{2} + \frac{M_1 - M_2}{l}$$

$$R_2 = V_2 = \frac{P}{2} - \frac{M_1 - M_2}{l}$$

$$M_3 \text{ (At center)} = \frac{Pl}{4} - \frac{M_1 + M_2}{2}$$

$$M_x \left(\text{When } x < \frac{l}{2}\right) = \left(\frac{P}{2} + \frac{M_1 - M_2}{l}\right)x - M_1$$

$$M_x \left(\text{When } x > \frac{l}{2}\right) = \frac{P}{2}(l - x) + \frac{(M_1 - M_2)x}{l} - M_1$$

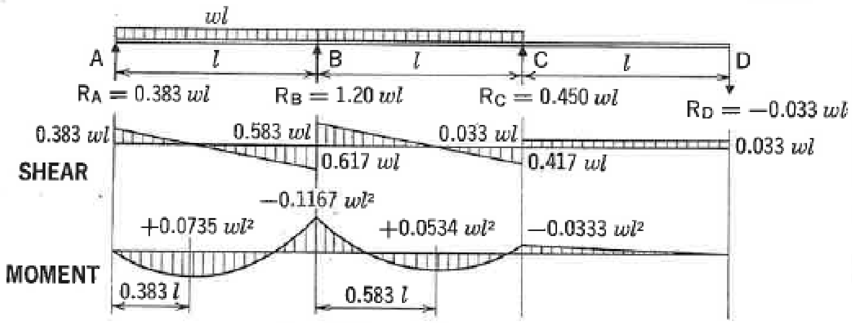
$$\Delta_x \left(\text{When } x < \frac{l}{2}\right) = \frac{Px}{48EI}\left(3l^2 - 4x^2 - \frac{8(l - x)}{Pl}\left[M_1(2l - x) + M_2(l + x)\right]\right)$$

# BEAM DIAGRAMS AND DEFLECTIONS

## For various static loading conditions

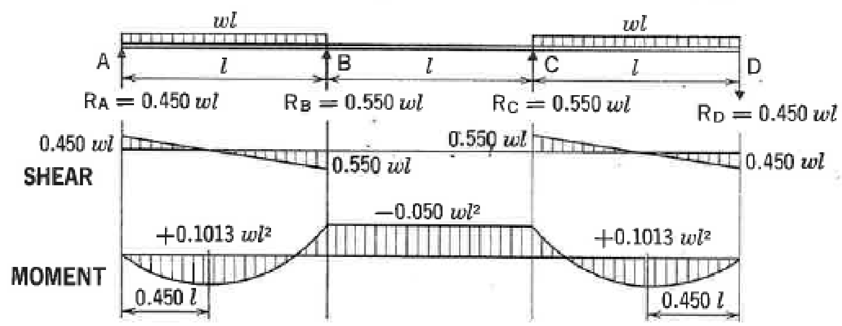
For meaning of symbols, see page 2 - 118.

34. CONTINUOUS BEAM—THREE EQUAL SPANS—ONE END SPAN UNLOADED



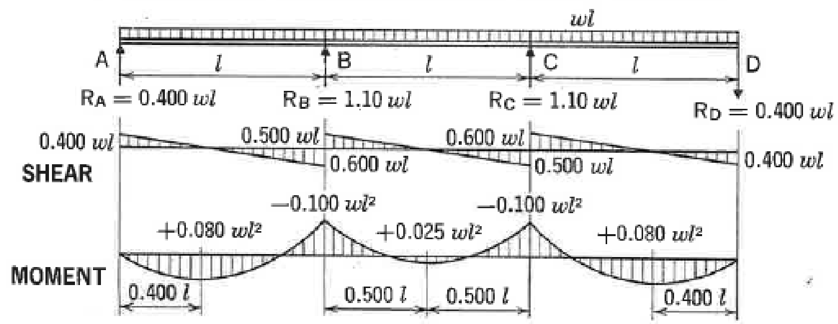

Δ Max. (0.430 $l$ from A) = 0.0059 $wl^4/EI$

35. CONTINUOUS BEAM—THREE EQUAL SPANS—END SPANS LOADED




Δ Max. (0.479 $l$ from A or D) = 0.0099 $wl^4/EI$

36. CONTINUOUS BEAM—THREE EQUAL SPANS—ALL SPANS LOADED




Δ Max. (0.446 $l$ from A or D) = 0.0069 $wl^4/EI$

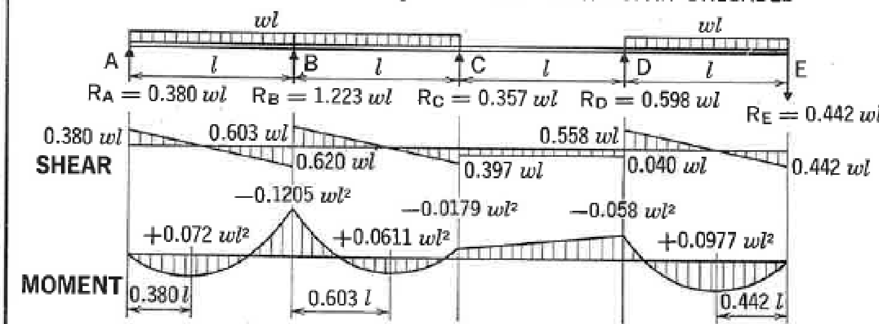


# BEAM DIAGRAMS AND DEFLECTIONS

## For various static loading conditions

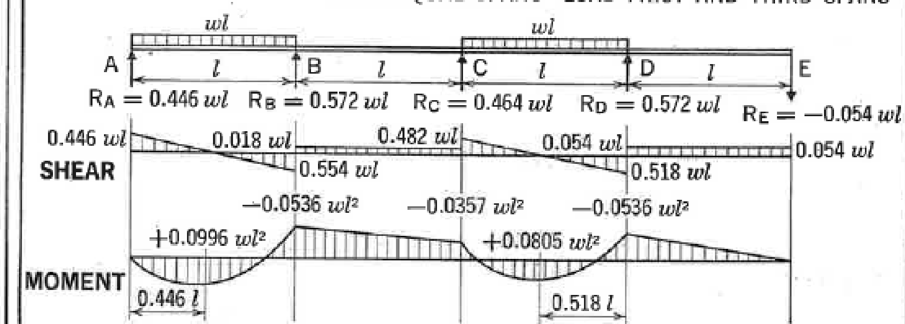
For meaning of symbols, see page 2 - 118.

37. CONTINUOUS BEAM—FOUR EQUAL SPANS—THIRD SPAN UNLOADED



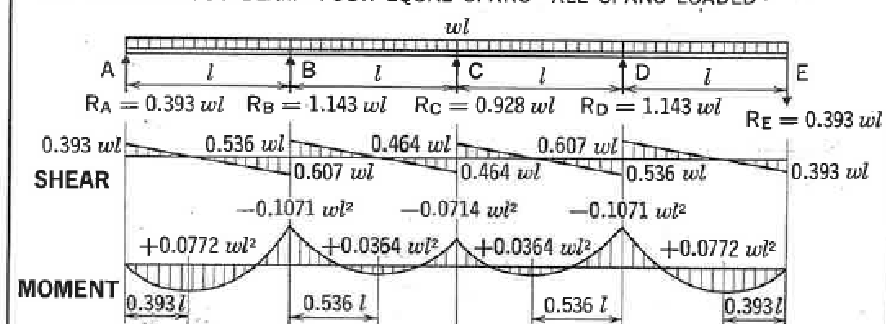

Δ Max. (0.475 $l$ from E) = 0.0094 $wl^4/EI$

38. CONTINUOUS BEAM—FOUR EQUAL SPANS—LOAD FIRST AND THIRD SPANS




Δ Max. (0.477 $l$ from A) = 0.0097 $wl^4/EI$

39. CONTINUOUS BEAM—FOUR EQUAL SPANS—ALL SPANS LOADED




Δ Max. (0.440 $l$ from A and E) = 0.0065 $wl^4/EI$