2.5 Weighted Residuals (Global, not FEA)

Consider the following model equation:

$$L(u) = \frac{d^2 u}{dx^2} + u + x = 0, \qquad x \in]0, 1[\tag{2.15}$$

with the essential boundary conditions $u = 0$ at $x = 0$ and $u = 0$ at $x = 1$ so that the exact solution is $u = Sin\,x / Sin\,1 - x$. We want to find a global approximate solution involving constants Δ_i, $1 \le i \le n$ that will lead to a set of n simultaneous equations. For homogeneous essential boundary conditions we usually pick a global product approximation of the form

$$u^* = g(x)\, f(x, \Delta_i) \tag{2.16}$$

where $g(x) \equiv 0$ on Γ. Here the boundary is $x = 0$ and $x - 1 = 0$ so we select a form such as

$$g_1(x) = x(1 - x)$$

or

$$g_2(x) = x - \frac{Sin\,x}{Sin\,1}.$$

We could pick $f(x, \Delta_i)$ as a polynomial

$$f(x) = \Delta_1 + \Delta_2 x + \cdots \Delta_n x^{(n-1)}.$$

For simplicity, select $n = 2$ and use $g_1(x)$ so the approximate solution is

$$u^*(x) = x(1 - x)(\Delta_1 + \Delta_2 x). \tag{2.17}$$

Here we will employ the method of *weighted residuals* to find the Δ's. From Eq. (2.17) we see that the residual error at any point is $R(x) = u'' + u + x$, or in expanded form:

$$R(x) = x + (-2 + x - x^2)\,\Delta_1 + (2 - 6x + x^2 - x^3)\,\Delta_2 \neq 0. \tag{2.18}$$

Note for future reference that the partial derivatives of the residual with respect to the unknown degrees of freedom are :

$$\frac{\partial R}{\partial \Delta_1} = (-2 + x - x^2), \qquad \frac{\partial R}{\partial \Delta_2} = (2 - 6x + x^2 - x^3).$$

The residual error will vanish everywhere only if we guess the exact solution. The method of weighted residuals requires that a weighted integral of the residual vanish, that is,

$$\int_0^1 R(x)\, w(x)\, dx \equiv 0 \tag{2.19}$$

where $w(x)$ is a weighting function. For an approximate solution with n constants we can split R into parts including and independent of the Δ_j, say

$$R = R_0 + \sum_{j=1}^{n} h_j(x)\, \Delta_j. \tag{2.20}$$

We use n weights to get the necessary algebraic equations

$$\int_\Omega R\, w_i\, d\Omega = \int_\Omega \left[R_0 + \sum_{j=1}^{n} h_j(x)\, \Delta_j \right] w_i\, d\Omega = 0_i, \quad 1 \le i \le n$$

or

$$\sum_{j=1}^{n} \int_\Omega h_j(x)\, w_i(x)\, \Delta_j\, d\Omega = -\int_\Omega R_0(x)\, w_i(x)\, d\Omega, \quad 1 \le i \le n. \tag{2.21}$$

In matrix form this system of equations is written as:

$$\underset{n \times n}{\mathbf{S}} \quad \underset{n \times 1}{\boldsymbol{\Delta}} \quad = \underset{n \times 1}{\mathbf{C}} . \qquad (2.22)$$

C) Galerkin Method: The concept here is to make the residual error orthogonal to the functions associated with the spatial influence of the constants. That is, let

$$u^*(x) = g(x)\, f(x, \Delta_i) = \sum_{i=1}^{n} h_i(x)\, \Delta_i .$$

Then for $n = 2$ and $h_1 = (x - x^2)$ and $h_2 = (x^2 - x^3)$, we set

$$w_i(x) \equiv h_i(x) \qquad (2.27)$$

so that we require

$$\int_0^1 R(x)\, h_1(x)\, dx = 0, \quad \int_0^1 R(x)\, h_2(x)\, dx = 0 \qquad (2.28)$$

so that Eq. (2.18) yields

$$\frac{3}{10}\Delta_1 + \frac{3}{20}\Delta_2 = \frac{1}{12}$$

$$\frac{3}{20}\Delta_1 + \frac{13}{105}\Delta_2 = \frac{1}{20}$$

which is again symmetric (for the self-adjoint equation). Solving gives degree of freedom values of $\Delta_1 = 71/369$, $\Delta_2 = 7/41$ and selected results at the three interior points of: 0.044, 0.070, and 0.060, respectively.

B) Least Squares Method: For the n equations pick

$$\int_0^1 R(x)\, w_i(x)\, dx = 0, \qquad 1 \le i \le n$$

with the weights defined as

$$w_i(x) = \frac{\partial R(x)}{\partial \Delta_i} . \qquad (2.25)$$

This is equivalent to

$$\int_0^1 R^2(x)\, dx \;\rightarrow\; \text{stationary (minimum)} . \qquad (2.26)$$

For this example

$$\int_0^1 R(x) \frac{\partial R}{\partial \Delta_1}\, dx = 0, \quad \int_0^1 R(x) \frac{\partial R}{\partial \Delta_2}\, dx = 0$$

and substitutions from Eq. (2.18) gives

$$\frac{202}{60}\Delta_1 + \frac{101}{60}\Delta_2 = \frac{55}{60}$$

$$\frac{101}{60}\Delta_1 + \frac{\cancel{1332}\ 393}{\cancel{60}\ 105}\Delta_2 = \frac{\cancel{393}\ 57}{60} .$$

It should be noted from Eq. (2.18) that this procedure yields a square matrix which is always symmetric. Solving gives $\Delta_1 = \cancel{0.192}$, $\Delta_2 = 0.\cancel{165}$ and selected results at the three interior points of: 0.043, 0.068, and 0.059, respectively. 0.1695

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